38 Let $\bullet$ be a two-operand infix operator (precedence 3) whose operands and result are of some type T. Let $\lozenge$ be a two-operand infix operator (precedence 7) whose operands are of type T and whose result is binary, defined by the axiom

$$a \lozenge b = a \bullet b = a$$

(a) Prove if $\bullet$ is idempotent then $\lozenge$ is reflexive.

(b) Prove if $\bullet$ is associative then $\lozenge$ is transitive.

(c) Prove if $\bullet$ is symmetric then $\lozenge$ is antisymmetric.

(d) If T is the binary values and $\bullet$ is $\wedge$, what is $\lozenge$?

(e) If T is the binary values and $\bullet$ is $\vee$, what is $\lozenge$?

(f) If T is the natural numbers and $\lozenge$ is $\leq$, what is $\bullet$?

(g) The axiom defines $\lozenge$ in terms of $\bullet$. Can it be inverted, so that $\bullet$ is defined in terms of $\lozenge$?

After trying the question, scroll down to the solution.

(a) Prove if $\bullet$ is idempotent then $\Diamond$ is reflexive.

§ $a \bullet a = a$ use axiom

$=$ $a \Diamond a$

(b) Prove if $\bullet$ is associative then $\Diamond$ is transitive.

§ $a \Diamond b \ \wedge \ b \Diamond c$ use axiom 2 times

$=$ $a \bullet b = a \ \wedge \ b \bullet c = b$ idempotence of $\wedge$

$=$ $a \bullet b = a \ \wedge \ a \bullet b = a \ \wedge \ b \bullet c = b$ use third conjunct to replace b in second

$=$ $a \bullet b = a \ \wedge \ a \bullet (b \bullet c) = a \ \wedge \ b \bullet c = b$ specialize: drop third conjunct

$\Rightarrow$ $a \bullet b = a \ \wedge \ a \bullet (b \bullet c) = a$ use associativity

$=$ $a \bullet b = a \ \wedge \ (a \bullet b) \bullet c = a$ use first conjunct to replace $a \bullet b$ in second

$=$ $a \bullet b = a \ \wedge \ a \bullet c = a$ specialize: drop first conjunct

$\Rightarrow$ $a \bullet c = a$ use axiom

$=$ $a \Diamond c$

(c) Prove if $\bullet$ is symmetric then $\Diamond$ is antisymmetric.

§ $a \Diamond b \ \wedge \ b \Diamond a$ use axiom 2 times

$=$ $a \bullet b = a \ \wedge \ b \bullet a = b$ use symmetry of $=$ and $\bullet$

$=$ $a = a \bullet b \ \wedge \ a \bullet b = b$ transitivity of $=$

$\Rightarrow$ $a = b$

(d) If T is the binary values and $\bullet$ is $\wedge$, what is $\Diamond$?

§ $a \Rightarrow b \ = \ a \wedge b = a$ so $\Diamond$ is $\Rightarrow$.

(e) If T is the binary values and $\bullet$ is $\vee$, what is $\Diamond$?

§ $a \Leftarrow b \ = \ a \vee b = a$ so $\Diamond$ is $\Leftarrow$.

(f) If T is the natural numbers and $\Diamond$ is $\leq$, what is $\bullet$?

§ $a \leq b \ = \ a \downarrow b = a$ so $\bullet$ is $\downarrow$.

(g) The axiom defines $\Diamond$ in terms of $\bullet$. Can it be inverted, so that $\bullet$ is defined in terms of $\Diamond$?

§ If T is the binary values we can invert as follows: $a \bullet b \ = \ a \Diamond b = a$. If T is anything else, we can invert under the assumption $a \Diamond b \ \vee \ b \Diamond a$. The inversion is

$a \bullet b \ =$ **if** $a \Diamond b$ **then** a **else** b **fi**