

390 For each of the following fixed-point equations, what does recursive construction yield?
Does it satisfy the fixed-point equation?

(a) $Q = \text{nat}_{\neg}(Q+3)$

(b) $D = 0, (D+1)_{\neg}(D-1)$

(c) $E = \text{nat}_{\neg}(E+1)$

(d) $F = 0, (\text{nat}_{\neg}F)+1$

After trying the question, scroll down to the solution.

$$\begin{aligned}
(a) \quad Q &= \text{nat}_{\neg}(Q+3) \\
\text{\S} \quad Q_0 &= \text{null} \\
Q_1 &= \text{nat}_{\neg}(\text{null}+3) = \text{nat}_{\neg}\text{null} = \text{nat} \\
Q_2 &= \text{nat}_{\neg}(\text{nat}+3) = 0, 1, 2 \\
Q_3 &= \text{nat}_{\neg}((0, 1, 2)+3) = \text{nat}_{\neg}(3, 4, 5) = 0, 1, 2, \text{nat}+6 \\
Q_4 &= \text{nat}_{\neg}((0, 1, 2, \text{nat}+6)+3) = \text{nat}_{\neg}(3, 4, 5, \text{nat}+9) = 0, 1, 2, 6, 7, 8 \\
Q_5 &= \text{nat}_{\neg}((0, 1, 2, 6, 7, 8)+3) = \text{nat}_{\neg}(3, 4, 5, 9, 10, 11) \\
&= 0, 1, 2, 6, 7, 8, \text{nat}+12
\end{aligned}$$

Time for a guess. It looks like there are two patterns: the even index pattern and the odd index pattern. So I guess

$$\begin{aligned}
Q_{2 \times n} &= 6 \times (0, \dots, n) + (0, \dots, 3) \\
Q_{2 \times n + 1} &= 6 \times (0, \dots, n) + (0, \dots, 3), (6 \times n, \dots, \infty)
\end{aligned}$$

From the even case, I propose

$$Q_{\infty} = 6 \times \text{nat} + (0, \dots, 3)$$

and now I have to check whether it satisfies the equation. Starting with the right side,

$$\begin{aligned}
&\text{nat}_{\neg}(Q_{\infty}+3) \\
&= \text{nat}_{\neg}(6 \times \text{nat} + (0, \dots, 3) + 3) \\
&= \text{nat}_{\neg}(6 \times \text{nat} + (3, \dots, 6)) && \text{here is an informal expansion} \\
&= \text{nat}_{\neg}((0, 6, 12, 18, 24, \dots) + (3, \dots, 6)) && \text{and an informal addition} \\
&= \text{nat}_{\neg}(3, 4, 5, 9, 10, 11, 15, 16, 17, 21, 22, 23, 27, 28, 29, \dots) \\
&= 0, 1, 2, 6, 7, 8, 12, 13, 14, 18, 19, 20, 24, 25, 26, \dots \\
&= (0, 6, 12, 18, 24, \dots) + (0, \dots, 3) \\
&= 6 \times \text{nat} + (0, \dots, 3) \\
&= Q_{\infty}
\end{aligned}$$

So it does satisfy the equation. From the odd case, I propose

$$Q_{\infty} = 6 \times \text{nat} + (0, \dots, 3), \infty$$

and now I have to check whether it satisfies the equation. Starting with the right side,

$$\begin{aligned}
&\text{nat}_{\neg}(Q_{\infty}+3) \\
&= \text{nat}_{\neg}((6 \times \text{nat} + (0, \dots, 3), \infty) + 3) \\
&= \text{nat}_{\neg}(6 \times \text{nat} + (3, \dots, 6), \infty) && \neg(\infty; \text{nat}) \\
&= \text{nat}_{\neg}(6 \times \text{nat} + (3, \dots, 6)) && \text{as before} \\
&= 6 \times \text{nat} + (0, \dots, 3) \\
&\neq 6 \times \text{nat} + (0, \dots, 3), \infty \\
&= Q_{\infty}
\end{aligned}$$

so this is not a solution.

$$\begin{aligned}
(b) \quad D &= 0, (D+1)_{\neg}(D-1) \\
\text{\S} \quad D_0 &= \text{null} \\
D_1 &= 0, (D_0+1)_{\neg}(D_0-1) \\
&= 0, (\text{null}+1)_{\neg}(\text{null}-1) \\
&= 0, \text{null}_{\neg}\text{null} \\
&= 0 \\
D_2 &= 0, (D_1+1)_{\neg}(D_1-1) \\
&= 0, (0+1)_{\neg}(0-1) \\
&= 0, 1_{\neg}-1 \\
&= 0, 1 \\
D_3 &= 0, (D_2+1)_{\neg}(D_2-1) \\
&= 0, ((0, 1)+1)_{\neg}((0, 1)-1) \\
&= 0, (1, 2)_{\neg}(-1, 0) \\
&= 0, 1, 2 \\
D_4 &= 0, (D_3+1)_{\neg}(D_3-1) \\
&= 0, ((0, 1, 2)+1)_{\neg}((0, 1, 2)-1)
\end{aligned}$$

$$\begin{aligned}
&= 0, (1, 2, 3) \bar{\neg} (-1, 0, 1) \\
&= 0, 2, 3 \\
D_5 &= 0, (D_4 + 1) \bar{\neg} (D_4 - 1) \\
&= 0, ((0, 2, 3) + 1) \bar{\neg} ((0, 2, 3) - 1) \\
&= 0, (1, 3, 4) \bar{\neg} (-1, 1, 2) \\
&= 0, 3, 4 \\
D_6 &= 0, (D_5 + 1) \bar{\neg} (D_5 - 1) \\
&= 0, ((0, 3, 4) + 1) \bar{\neg} ((0, 3, 4) - 1) \\
&= 0, (1, 4, 5) \bar{\neg} (-1, 2, 3) \\
&= 0, 1, 4, 5 \\
D_7 &= 0, (D_6 + 1) \bar{\neg} (D_6 - 1) \\
&= 0, ((0, 1, 4, 5) + 1) \bar{\neg} ((0, 1, 4, 5) - 1) \\
&= 0, (1, 2, 5, 6) \bar{\neg} (-1, 0, 3, 4) \\
&= 0, 1, 2, 5, 6 \\
D_8 &= 0, (D_7 + 1) \bar{\neg} (D_7 - 1) \\
&= 0, ((0, 1, 2, 5, 6) + 1) \bar{\neg} ((0, 1, 2, 5, 6) - 1) \\
&= 0, (1, 2, 3, 6, 7) \bar{\neg} (-1, 0, 1, 4, 5) \\
&= 0, 2, 3, 6, 7
\end{aligned}$$

It's still hard to see the patterns, so maybe we have to go a bit farther. Then we see

$$\begin{aligned}
D_{4 \times n + 1} &= 0, 4 \times (0, \dots, n) + (3, 4) \\
D_{4 \times n + 2} &= 0, 1, 4 \times (0, \dots, n) + (4, 5) \\
D_{4 \times n + 3} &= 0, 1, 2, 4 \times (0, \dots, n) + (5, 6) \\
D_{4 \times n + 4} &= 0, 2, 3, 4 \times (0, \dots, n) + (6, 7)
\end{aligned}$$

We have a choice of four possible answers for D_∞ , but none of them satisfies the equation. Recursive construction fails.

$$\begin{aligned}
(c) \quad E &= \text{nat} \bar{\neg} (E + 1) \\
\text{\S} \quad E_0 &= \text{null} \\
E_1 &= \text{nat} \\
E_2 &= 0 \\
E_3 &= 0, \text{nat} + 2 \\
E_4 &= 0, 2 \\
E_5 &= 0, 2, \text{nat} + 4 \\
E_{2 \times n} &= 2 \times (0, \dots, n) \\
E_{2 \times n + 1} &= 2 \times (0, \dots, n), \text{nat} + 2 \times n
\end{aligned}$$

From the even case, we propose

$$E_\infty = 2 \times \text{nat}$$

which satisfies the equation. From the odd case, we propose

$$E_\infty = 2 \times \text{nat}, \infty$$

which does not satisfy the equation.

$$\begin{aligned}
(d) \quad F &= 0, (\text{nat} \bar{\neg} F) + 1 \\
\text{\S} \quad F_0 &= \text{null} \\
F_1 &= \text{nat} \\
F_2 &= 0 \\
F_3 &= 0, \text{nat} + 2 \\
F_4 &= 0, 2 \\
F_5 &= 0, 2, \text{nat} + 4 \\
F_{2 \times n} &= 2 \times (0, \dots, n) \\
F_{2 \times n + 1} &= 2 \times (0, \dots, n), \text{nat} + 2 \times n
\end{aligned}$$

From the even case, we propose

$$F_{\infty} = 2 \times nat$$

which satisfies the fixed-point equation. From the odd case, we propose

$$F_{\infty} = 2 \times nat, \infty$$

which does not satisfy the fixed-point equation.