401 (bit strings) Define val: $*(0, 1) \to nat$ as follows:

$$val\ nil = 0$$
$$val\ (a; 0) = 2 \times val\ a$$
$$val\ (a; 1) = 2 \times val\ a + 1$$

Prove

(a) $val\ 0 = 0$

(b) $val\ 1 = 1$

(c) $val\ (0; a) = val\ a$

(d) $val\ (1; a) = 2^{\leftrightarrow a} + val\ a$

(e) $val\ (a; b) = val\ a \times 2^{\leftrightarrow b} + val\ b$

After trying the question, scroll down to the start of a solution.

(a) $val\ 0 = 0$

§ $\top$ axiom

$=$ $val\ (a; 0) = 2 \times val\ a$ specialize

$=$ $val\ (nil; 0) = 2 \times val\ nil$ string theory, and axiom

$=$ $\text{val}\ 0 = 2 \times 0$ arithmetic

$=$ 0

(b) $val\ 1 = 1$

§ $\top$ axiom

$=$ $val\ (a; 1) = 2 \times val\ a + 1$ specialize

$=$ $val\ (nil; 1) = 2 \times val\ nil + 1$ string theory, and axiom

$=$ $\text{val}\ 1 = 2 \times 0 + 1$ arithmetic

$=$ 1

(c) $val\ (0; a) = val\ a$

§ by induction on the form of a. It cannot be fully formal because we don't have the induction axiom for * (see Exercise 397(a)). Informally, a is either nil or $b;0$ or $b;1$ for some b: $*(0, 1)$. The base case is nil. The induction hypothesis for the other two cases is $val\ (0; b) = val\ b$.

Case $a = nil$.

$val\ (0; a)$ $a=nil$

$=$ $val\ (0; nil)$ string theory

$=$ $val\ 0$ part (a)

$=$ 0 axiom

$=$ $val\ nil$

Case $a = b;0$

$val\ (0; a)$ $a = b;0$

$=$ $val\ (0; b; 0)$ axiom

$=$ $2 \times val\ (0; b)$ induction hypothesis

$=$ $2 \times val\ b$ axiom

$=$ $val\ (b; 0)$ $a = b;0$

$=$ $val\ a$

Case $a = b;1$ just like case $a = b;0$.

(d) $val\ (1; a) = 2^{\leftrightarrow a} + val\ a$

§ UNFINISHED

(e) $val\ (a; b) = val\ a \times 2^{\leftrightarrow b} + val\ b$

§ UNFINISHED