408 Let i be an integer variable. Add recursive time. Any way you can, find a fixed-point of

(a) $walk$ = **if** $i\ge 0$ **then** $i:= i-2$. $walk$. $i:= i+1$. $walk$. $i:= i+1$ **else** ok **fi**

(b) $crawl$ = **if** $i\ge 0$ **then** $i:= i-1$. $crawl$. $i:= i+2$. $crawl$. $i:= i-1$ **else** ok **fi**

(c) run = **if** $even\ i$ **then** $i:= i/2$ **else** $i:= i+1$ **fi**. **if** $i=1$ **then** ok **else** run **fi**

After trying the question, scroll down to the solution.

(a) $\quad walk \;=\; \mathbf{if}\; i\ge 0\; \mathbf{then}\; i:= i-2.\; walk.\; i:= i+1.\; walk.\; i:= i+1\; \mathbf{else}\; ok\; \mathbf{fi}$

§ Putting $t:= t+1$ before just the first call of $walk$ is enough, though we could put it before both calls. If the $=$ had been $\Leftarrow$, we could prove $i'=i \;\wedge\; t' \le t+2^i$ as follows. Proof by cases:

$i\ge 0 \;\wedge\; (i:= i-2.\; i'=i \;\wedge\; t' \le t+2^i.\; i:= i+1.\; i'=i \;\wedge\; t' \le t+2^i.\; i:= i+1)$ $\qquad$ substitution

$=\;\; i\ge 0 \;\wedge\; (i'=i-2 \;\wedge\; t' \le t+2^{i-2}.\; i'=i+1 \;\wedge\; t' \le t+2^{i+1}.\; i:= i+1)$

$=\;\; i\ge 0 \;\wedge\; \exists i'', t'', i''', t'''\cdot\; i''=i-2 \wedge t''\le t+2^{i-2} \wedge i'''=i''+1 \wedge t'''\le t''+2^{i''+1} \wedge i'=i'''+1$

$=\;\; i\ge 0 \;\wedge\; i'=i \;\wedge\; t' \le t+2^{i-2}+2^{i-1}$

$=\;\; i\ge 0 \;\wedge\; i'=i \;\wedge\; t' \le t+(3/4)\times 2^i$

$\Rightarrow\;\; i'=i \;\wedge\; t' \le t+2^i$

$i<0 \wedge ok$

$\Rightarrow\;\; i'=i \;\wedge\; t' \le t+2^i$

But the question did not say $\Leftarrow$, so I haven't found the fixed-point asked for.

(b) $\quad crawl \;=\; \mathbf{if}\; i\ge 0\; \mathbf{then}\; i:= i-1.\; crawl.\; i:= i+2.\; crawl.\; i:= i-1\; \mathbf{else}\; ok\; \mathbf{fi}$

§ Putting $t:= t+1$ before just the first call of $crawl$ is enough, though we could put it before both calls. Here are two answers.

$\mathbf{if}\; i\ge 0\; \mathbf{then}\; t'=\infty\; \mathbf{else}\; ok\; \mathbf{fi}$

$\mathbf{if}\; i\ge 0\; \mathbf{then}\; t:= \infty\; \mathbf{else}\; ok\; \mathbf{fi}$

I'll check the second one.

$\mathbf{if}\; i\ge 0\; \mathbf{then}\; i:= i-1.\; t:= t+1.\; \mathbf{if}\; i\ge 0\; \mathbf{then}\; t:= \infty\; \mathbf{else}\; ok\; \mathbf{fi}.$
$\qquad i:= i+2.\; \mathbf{if}\; i\ge 0\; \mathbf{then}\; t:= \infty\; \mathbf{else}\; ok\; \mathbf{fi}.$
$\qquad i:= i-1$ $\qquad$ distribute $i:= i-1$ back into previous **if**
$\mathbf{else}\; ok\; \mathbf{fi}$

$=\;\; \mathbf{if}\; i\ge 0\; \mathbf{then}\; i:= i-1.\; t:= t+1.\; \mathbf{if}\; i\ge 0\; \mathbf{then}\; t:= \infty\; \mathbf{else}\; ok\; \mathbf{fi}.$
$\qquad i:= i+2.\; \mathbf{if}\; i\ge 0\; \mathbf{then}\; t:= \infty.\; i:= i-1\; \mathbf{else}\; ok.\; i:= i-1\; \mathbf{fi}$
$\mathbf{else}\; ok\; \mathbf{fi}$ $\qquad$ ok is identity. Also, expand $i:= i-1$ twice

$=\;\; \mathbf{if}\; i\ge 0\; \mathbf{then}\; i:= i-1.\; t:= t+1.\; \mathbf{if}\; i\ge 0\; \mathbf{then}\; t:= \infty\; \mathbf{else}\; ok\; \mathbf{fi}.$
$\qquad i:= i+2.\; \mathbf{if}\; i\ge 0\; \mathbf{then}\; t:= \infty.\; i'=i-1 \wedge t'=t\; \mathbf{else}\; i'=i-1 \wedge t'=t\; \mathbf{fi}$
$\mathbf{else}\; ok\; \mathbf{fi}$ $\qquad$ substitution law using $t:= \infty$

$=\;\; \mathbf{if}\; i\ge 0\; \mathbf{then}\; i:= i-1.\; t:= t+1.\; \mathbf{if}\; i\ge 0\; \mathbf{then}\; t:= \infty\; \mathbf{else}\; ok\; \mathbf{fi}.$ $\qquad$ expand $t:= \infty$ and ok
$\qquad i:= i+2.\; \mathbf{if}\; i\ge 0\; \mathbf{then}\; i'=i-1 \wedge t'=\infty\; \mathbf{else}\; i'=i-1 \wedge t'=t\; \mathbf{fi}$
$\mathbf{else}\; ok\; \mathbf{fi}$

$=\;\; \mathbf{if}\; i\ge 0\; \mathbf{then}\; i:= i-1.\; t:= t+1.\; \mathbf{if}\; i\ge 0\; \mathbf{then}\; i'=i \wedge t'=\infty\; \mathbf{else}\; i'=i \wedge t'=t\; \mathbf{fi}.$ subst law twice
$\qquad i:= i+2.\; \mathbf{if}\; i\ge 0\; \mathbf{then}\; i'=i-1 \wedge t'=\infty\; \mathbf{else}\; i'=i-1 \wedge t'=t\; \mathbf{fi}$ $\qquad$ substitution law
$\mathbf{else}\; ok\; \mathbf{fi}$

$=\;\; \mathbf{if}\; i\ge 0\; \mathbf{then}\; \mathbf{if}\; i\ge 1\; \mathbf{then}\; i'=i-1 \wedge t'=\infty\; \mathbf{else}\; i'=i-1 \wedge t'=t+1\; \mathbf{fi}.$ $\qquad$ factor (distributive law)
$\qquad \mathbf{if}\; i\ge -2\; \mathbf{then}\; i'=i+1 \wedge t'=\infty\; \mathbf{else}\; i'=i+1 \wedge t'=t\; \mathbf{fi}$ $\qquad$ factor (distributive law)
$\mathbf{else}\; ok\; \mathbf{fi}$

$=\;\; \mathbf{if}\; i\ge 0\; \mathbf{then}\; \mathbf{if}\; i\ge 1\; \mathbf{then}\; t'=\infty\; \mathbf{else}\; t'=t+1\; \mathbf{fi} \wedge i'=i-1.$ $\qquad$ definition of .
$\qquad \mathbf{if}\; i\ge -2\; \mathbf{then}\; t'=\infty\; \mathbf{else}\; t'=t\; \mathbf{fi} \wedge i'=i+1$
$\mathbf{else}\; ok\; \mathbf{fi}$

$=\;\; \mathbf{if}\; i\ge 0\; \mathbf{then}\; \exists i'', t''\cdot\;\; \mathbf{if}\; i\ge 1\; \mathbf{then}\; t''=\infty\; \mathbf{else}\; t''=t+1\; \mathbf{fi} \wedge i''=i-1$ $\qquad$ one-point on i''
$\qquad \wedge\; \mathbf{if}\; i''\ge -2\; \mathbf{then}\; t'=\infty\; \mathbf{else}\; t'=t''\; \mathbf{fi} \wedge i'=i''+1$
$\mathbf{else}\; ok\; \mathbf{fi}$

$=\;\; \mathbf{if}\; i\ge 0\; \mathbf{then}\; \exists t''\cdot\;\; \mathbf{if}\; i\ge 1\; \mathbf{then}\; t''=\infty\; \mathbf{else}\; t''=t+1\; \mathbf{fi}$
$\qquad \wedge\; \mathbf{if}\; i\ge -1\; \mathbf{then}\; t'=\infty\; \mathbf{else}\; t'=t''\; \mathbf{fi} \wedge i'=i$ $\qquad$ context $i\ge 0$
$\mathbf{else}\; ok\; \mathbf{fi}$

$=\;\; \mathbf{if}\; i\ge 0\; \mathbf{then}\; \exists t''\cdot\;\; \mathbf{if}\; i\ge 1\; \mathbf{then}\; t''=\infty\; \mathbf{else}\; t''=t+1\; \mathbf{fi}$
$\qquad \wedge\; t'=\infty \wedge i'=i$
$\mathbf{else}\; ok\; \mathbf{fi}$ $\qquad$ Now a tricky move. In the inner **then**-part, use context $i\ge 1$.

And in the inner **else**-part, use context $i<1$.

$=$ **if** $i\ge 0$ **then** $\exists t''\cdot$ **if** $i\ge 1$ **then** t'' = **if** $i\ge 1$ **then** ∞ **else** $t+1$ **fi**
else t''= **if** $i\ge 1$ **then** ∞ **else** $t+1$ **fi fi**
$\wedge\ t'=\infty\ \wedge\ i'=i$
else *ok* **fi** generic case idempotent

$=$ **if** $i\ge 0$ **then** $\exists t''\cdot$ t'' = **if** $i\ge 1$ **then** ∞ **else** $t+1$ **fi**
$\wedge\ t'=\infty\ \wedge\ i'=i$
else *ok* **fi** one-point for t''

$=$ **if** $i\ge 0$ **then** $t'=\infty\ \wedge\ i'=i$
else *ok* **fi** assignment

$=$ **if** $i\ge 0$ **then** $t:=\infty$ **else** *ok* **fi**

So it's a fixed-point.

(c) *run* $=$ **if** *even i* **then** $i:=i/2$ **else** $i:=i+1$ **fi**.
if $i=1$ **then** *ok* **else** *run* **fi**

§ Without adding time, $i'=1$ and $i\ge 1\Rightarrow i'=1$ are fixed-points. With time, it's difficult to say a fixed-point since it requires saying the exact execution time. If we had an implication instead of an equation, we could get away with a time bound. Recursive construction just leads to a mess, and isn't helpful. To state the exact execution time, define

$f\ =\ \langle i\colon int\cdot$ **if** $i=2$ **then** 0 **else if** *even i* **then** $1+f(i/2)$ **else** $1+f(i+1)$ **fi fi**$\rangle$

Now we can find the following two fixed-points:

$i'=1\ \ \wedge\ \ t'=t+f\,i$

$(i\ge 1\Rightarrow i'=1)\ \ \wedge\ \ t'=t+f\,i$

although f seems like an unfair trick.