432 From the axioms of program queue theory (Subsection 7.1.4), prove

(a) $\mathit{front}'=3 \Leftarrow \mathit{mkemptyq}.\ \mathit{join}\ 3$

(b) $\mathit{front}'=4 \Leftarrow \mathit{mkemptyq}.\ \mathit{join}\ 3.\ \mathit{join}\ 4.\ \mathit{leave}$

After trying the question, scroll down to the solution.

§ The axioms of program Queue Theory are

(0) $isemptyq' \Leftarrow mkemptyq$

(1) $isemptyq \Rightarrow front'=x \wedge \neg isemptyq' \Leftarrow join\ x$

(2) $\neg isemptyq \Rightarrow front'=front \wedge \neg isemptyq' \Leftarrow join\ x$

(3) $isemptyq \Rightarrow (join\ x.\ leave = mkemptyq)$

(4) $\neg isemptyq \Rightarrow (join\ x.\ leave = leave.\ join\ x)$

(a) $front'=3 \Leftarrow mkemptyq.\ join\ 3$

§ $mkemptyq.\ join\ 3$ — use (0) and (1) and monotonicity of .

$\Rightarrow$ $isemptyq'.\ isemptyq \Rightarrow front'=3 \wedge \neg isemptyq'$ — use definition of .

$=$ $\exists isemptyq'', front''\cdot\ isemptyq'' \wedge (isemptyq'' \Rightarrow front'=3 \wedge \neg isemptyq')$ — discharge

$=$ $\exists isemptyq'', front''\cdot\ isemptyq'' \wedge front'=3 \wedge \neg isemptyq'$ — specialization and monotonicity of $\exists$

$\Rightarrow$ $\exists isemptyq'', front''\cdot\ front'=3$ — unused quantifier

$=$ $front'=3$

(b) $front'=4 \Leftarrow mkemptyq.\ join\ 3.\ join\ 4.\ leave$

§ Plan: Use (4) to commute ($join\ 4.\ leave$) . Then use (3) to change ($join\ 3.\ leave$) into $mkemptyq$. Then make ($mkemptyq.\ mkemptyq$) into $mkemptyq$. Then use (0) and (1) to make ($mkemptyq.\ join\ 4$) into $front'=4$.

(5) $\top$ — (1)

$=$ $(isemptyq \Rightarrow front'=x \wedge \neg isemptyq' \Leftarrow join\ x)$ — portation

$=$ $join\ x \wedge isemptyq \Rightarrow front'=x \wedge \neg isemptyq'$ — specialize

$\Rightarrow$ $join\ x \wedge isemptyq \Rightarrow \neg isemptyq'$

(6) $\top$ — (2)

$=$ $(\neg isemptyq \Rightarrow front'=front \wedge \neg isemptyq' \Leftarrow join\ x)$ — portation

$=$ $join\ x \wedge \neg isemptyq \Rightarrow front'=front \wedge \neg isemptyq'$ — specialize

$\Rightarrow$ $join\ x \wedge \neg isemptyq \Rightarrow \neg isemptyq'$

(7) $\top$ — (5) and (6)

$=$ $(join\ x \wedge isemptyq \Rightarrow \neg isemptyq') \wedge (join\ x \wedge \neg isemptyq \Rightarrow \neg isemptyq')$ — antidistributive

$=$ $join\ x \wedge isemptyq \vee join\ x \wedge \neg isemptyq \Rightarrow \neg isemptyq'$ — distributive

$=$ $join\ x \wedge (isemptyq \vee \neg isemptyq) \Rightarrow \neg isemptyq'$ — excluded middle and identity

$=$ $join\ x \Rightarrow \neg isemptyq'$ — inclusion

$=$ $(join\ x = join\ x \wedge \neg isemptyq')$

(9) $a \Rightarrow (b=c)$ — antisymmetry

$=$ $a \Rightarrow (b \Rightarrow c) \wedge (c \Rightarrow b)$ — distributive

$=$ $(a \Rightarrow (b \Rightarrow c)) \wedge (a \Rightarrow (c \Rightarrow b))$ — specialize

$\Rightarrow$ $a \Rightarrow (b \Rightarrow c)$ — portation

$=$ $a \wedge b \Rightarrow c$ — symmetry

$=$ $b \wedge a \Rightarrow c$ — portation

$=$ $b \Rightarrow (a \Rightarrow c)$

(10) $\top$ — (4)

$=$ $\neg isemptyq \Rightarrow (join\ x.\ leave = leave.\ join\ x)$ — (9)

$\Longrightarrow$ $(join\ x.\ \ leave) \Rightarrow (\neg isemptyq \Rightarrow (leave.\ \ join\ x))$

(11) $\top$ (3)

$\Longrightarrow$ $(join\ x.\ \ leave) \Rightarrow (isemptyq \Rightarrow mkemptyq)$

Now the main proof.

	$mkemptyq.\ \ join\ 3.\ \ join\ 4.\ \ leave$	use (10) and monotonicity of .
$\Longrightarrow$	$mkemptyq.\ \ join\ 3.\ \ \neg isemptyq \Rightarrow (leave.\ \ join\ 4)$	use (8) and monotonicity of .
$\Longrightarrow$	$mkemptyq.\ \ join\ 3 \wedge \neg isemptyq'.\ \ \neg isemptyq \Rightarrow (leave.\ \ join\ 4)$	condition law
$\Longrightarrow$	$mkemptyq.\ \ join\ 3.\ \ leave.\ \ join\ 4$	use (11)
$\Longrightarrow$	$mkemptyq.\ \ isemptyq \Rightarrow mkemptyq.\ \ join\ 4$	use (0) twice
$\Longrightarrow$	$isemptyq'.\ \ isemptyq \Rightarrow isemptyq'.\ \ join\ 4$	use definition of . on first .
$=$	$(\exists isemptyq'', front''\cdot\ \ isemptyq'' \wedge (isemptyq'' \Rightarrow isemptyq')).\ \ join\ 4$	discharge
$=$	$(\exists isemptyq'', front''\cdot\ \ isemptyq'' \wedge isemptyq').\ \ join\ 4$	one-pt and unused
$=$	$isemptyq'.\ \ join\ 4$	use (1)
$=$	$isemptyq'.\ \ isemptyq \Rightarrow front'{=}4 \wedge \neg isemptyq'$	use definition of .
$=$	$\exists isemptyq'', front''\cdot\ \ isemptyq'' \wedge (isemptyq'' \Rightarrow front'{=}4 \wedge \neg isemptyq')$	discharge
$=$	$\exists isemptyq'', front''\cdot\ \ isemptyq'' \wedge front'{=}4 \wedge \neg isemptyq'$	one-pt and unused
$=$	$front'{=}4 \wedge \neg isemptyq'$	specialize
$=$	$front'{=}4$	

I hope there's a shorter proof.