47 Let $\otimes$ be a two-operand infix operator (precedence 3) with natural operands and an extended natural result. Informally, $n \otimes m$ means "the number of times that n is a factor of m". It is defined by the following two axioms.

$$m{:}\ n \times nat \ \vee\ n \otimes m = 0$$

$$n \neq 0 \ \Rightarrow\ n \otimes (m \times n) \ =\ n \otimes m + 1$$

(a) Make a 3×3 chart of the values of $(0,..3) \otimes (0,..3)$.

(b) Show that the axioms become inconsistent if the antecedent of the second axiom is removed.

(c) How should we change the axioms to allow $\otimes$ to have extended natural operands?

After trying the question, scroll down to the solution.

(a) Make a 3×3 chart of the values of $(0,..3)\otimes(0,..3)$.

§

	0	1	2
0		0	0
1	∞	∞	∞
2	∞	0	1

(b) Show that the axioms become inconsistent if the antecedent of the second axiom is removed.

§

	$\top$	axiom
=	$m{:}\ n\times nat \ \vee\ n\otimes m = 0$	specialization
=	$1{:}\ 0\times nat \ \vee\ 0\otimes 1 = 0$	$1{:}\ 0\times nat$ is $\perp$
=	$0\otimes 1 = 0$	

	$0\otimes 0$	$0 = 1\times 0$
=	$0\otimes(1\times 0)$	use $n\otimes(m\times n) = n\otimes m + 1$
=	$0\otimes 1 + 1$	use previous result
=	$0+1$	
=	1	

	$0\otimes 0$	$0 = 0\times 0$
=	$0\otimes(0\times 0)$	use $n\otimes(m\times n) = n\otimes m + 1$
=	$0\otimes 0 + 1$	use previous result
=	$1+1$	
=	2	

Hence $1 = 2$.

(c) How should we change the axioms to allow $\otimes$ to have extended natural operands?

§ From the first axiom, instantiating with $m{=}\infty$ and $n{=}1$, we get

	$\infty{:}\ 1\times nat \ \vee\ 1\otimes\infty = 0$	
=	$\perp \ \vee\ 1\otimes\infty = 0$	
=	$1\otimes\infty = 0$	

From the second axiom, instantiating with $m{=}\infty$ and $n{=}1$, we get

	$1\neq 0 \ \Rightarrow\ 1\otimes(\infty\times 1) = 1\otimes\infty + 1$	
=	$1\otimes\infty = 1\otimes\infty + 1$	now use what we got from the first axiom
=	$0 = 0+1$	

So we can't leave the axioms as they are. We can change *nat* to *xnat* in the first axiom; now for $n\neq 0$ we have $n\otimes\infty = \infty$. Perhaps we don't want $\infty\otimes\infty = \infty$, so perhaps we should weaken the second axioms to $0<n<\infty \ \Rightarrow\ n\otimes(m\times n) = n\otimes m + 1$. We now have no answer for $\infty\otimes m$, which should be 0 for finite m .