510 From the fixed-point equation

$$twos \;=\; c!\,2.\;\; t{:=}\,t{+}1.\;\; twos$$

use recursive construction to find

(a) the weakest fixed-point.

(b) a strongest implementable fixed-point.

(c) the strongest fixed-point.

After trying the question, scroll down to the solution.

§ $twos \;=\; \mathcal{M}_{\boldsymbol{w}}=2 \land \mathcal{T}_{\boldsymbol{w}}=t \land \boldsymbol{w}'=\boldsymbol{w}+1 \land \boldsymbol{r}'=\boldsymbol{r} \land t'=t+1.\; twos$

(a) the weakest fixed-point.

§ If we start with $\top$, then

$$twos_n \;=\; \forall i{:}\; 0,..n\cdot \mathcal{M}_{\boldsymbol{w}+i}=2 \land \mathcal{T}_{\boldsymbol{w}+i}=t+i$$
$$twos_\infty \;=\; \forall i{:}\; nat\cdot \mathcal{M}_{\boldsymbol{w}+i}=2 \land \mathcal{T}_{\boldsymbol{w}+i}=t+i$$

(b) a strongest implementable fixed-point.

§

$$twos \;=\; \mathcal{M}_{\boldsymbol{w}}=2 \land \mathcal{T}_{\boldsymbol{w}}=t \land \boldsymbol{w}'=\boldsymbol{w}+1 \land \boldsymbol{r}'=\boldsymbol{r} \land t'=t+1.\; twos$$
$$twos_0 \;=\; \boldsymbol{w}'\ge\boldsymbol{w} \land \boldsymbol{r}'\ge\boldsymbol{r} \land t'\ge t$$
$$twos_1 \;=\; \mathcal{M}_{\boldsymbol{w}}=2 \land \mathcal{T}_{\boldsymbol{w}}=t \land \boldsymbol{w}'\ge\boldsymbol{w}+1 \land \boldsymbol{r}'\ge\boldsymbol{r} \land t'\ge t+1$$
$$twos_2 \;=\; \mathcal{M}_{\boldsymbol{w}}=2 \land \mathcal{T}_{\boldsymbol{w}}=t \land \mathcal{M}_{\boldsymbol{w}+1}=2 \land \mathcal{T}_{\boldsymbol{w}+1}=t+1 \land \boldsymbol{w}'\ge\boldsymbol{w}+2 \land \boldsymbol{r}'\ge\boldsymbol{r} \land t'\ge t+2$$
$$twos_3 \;=\; \mathcal{M}_{\boldsymbol{w}}=2 \land \mathcal{T}_{\boldsymbol{w}}=t \land \mathcal{M}_{\boldsymbol{w}+1}=2 \land \mathcal{T}_{\boldsymbol{w}+1}=t+1 \land \mathcal{M}_{\boldsymbol{w}+2}=2 \land \mathcal{T}_{\boldsymbol{w}+2}=t+2 \land \boldsymbol{w}'\ge\boldsymbol{w}+3 \land \boldsymbol{r}'\ge\boldsymbol{r} \land t'\ge t+3$$
$$twos_n \;=\; (\forall i{:}\; 0,..n\cdot \mathcal{M}_{\boldsymbol{w}+i}=2 \land \mathcal{T}_{\boldsymbol{w}+i}=t+i) \land \boldsymbol{w}'\ge\boldsymbol{w}+n \land \boldsymbol{r}'\ge\boldsymbol{r} \land t'\ge t+n$$
$$twos_\infty \;=\; (\forall i{:}\; nat\cdot \mathcal{M}_{\boldsymbol{w}+i}=2 \land \mathcal{T}_{\boldsymbol{w}+i}=t+i) \land \boldsymbol{w}'=\infty \land \boldsymbol{r}'=\boldsymbol{r} \land t'=\infty$$

This is both implementable and deterministic, so it is a strongest implementable fixed-point.

(c) the strongest fixed-point.

§ If we start with $\bot$, then

$$twos_n \;=\; twos_\infty \;=\; \bot$$