54 (Cantor's paradise) Show that $¢ϟB > ¢B$ is neither a theorem nor an antitheorem.

After trying the question, scroll down to the solution.

§ ¢ϟ*nat* > ¢*nat* — bunch interval law $nat = 0,..\infty$
= ¢ϟ*nat* > ¢(0,..∞) — bunch interval law ¢(*x*,..*y*) = *y*–*x*
= ¢ϟ*nat* > ∞–0 — absorption or identity
= ¢ϟ*nat* > ∞ — conjoin Extremes Law
= ¢ϟ*nat* > ∞ ∧ –∞ ≤ *x* ≤ ∞ — specialize
⇒ ¢ϟ*nat* > ∞ ∧ ¢ϟ*nat* ≤ ∞ — generic totality
= ¬ ¢ϟ*nat* ≤ ∞ ∧ ¢ϟ*nat* ≤ ∞ — noncontradiction
= ⊥

From that we conclude that ¢ϟ*B* > ¢*B* is not a theorem.

¢ϟ*null* > ¢*null*
= ¢{*null*} > ¢*null*
= $1 > 0$
= ⊤

From that we conclude that ¢ϟ*B* > ¢*B* is not an antitheorem.

If we were to add ¢ϟ*B* > ¢*B* as an axiom or as an antiaxiom we would cause inconsistency. We could add

¢*B* < ∞ ⇒ ¢ϟ*B* > ¢*B*

as an axiom without causing inconsistency.

The mathematics presented in the textbook was designed for the purpose of describing and reasoning about computation, including execution time. Execution time can be infinite, so ∞ is included in our number theory. We have no use for many infinite numbers, so they are not included in our number theory.

Mathematicians generally do not have the Extremes Law $-\infty \le x \le \infty$, and they do have a law similar to ¢ϟ*B* > ¢*B* . So they have an infinite number of infinite numbers. I have no idea what applications that mathematics is designed for. Most mathematicians do not think that mathematics is designed; they think there are mathematical facts that are independent of people (those mathematical facts would be facts even if there were no people), and people just discover those facts. They think that people have discovered that ¢ϟ*B* > ¢*B* is a fact. But I think that mathematics is a product of human design, and we should be designing math to fit applications. See the Size of a Set.

For describing computation, we have no use for $-\infty$. But we have a use for ∞ , and we have a use for numeric negation – , so we get $-\infty$ whether we want it or not. We also have the equivalent of many infinite numbers, and infinitesimal numbers too, whether we want them of not. Let x be real, and let p be positive real. Then $0 < 0;x < p$. Hence the string of numbers $0;x$ is like an infinitesimal, larger than 0 but smaller than any positive real. And $0 < 0;0;x < 0;p < p$, so $0;0;x$ is like an infinitesimal smaller than $0;p$. And so on. Similarly $x < \infty < \infty;x < \infty;\infty < \infty;\infty;x$ and so on. So strings of numbers starting with ∞ are like infinities of different sizes.