

6 Prove each of the following laws of Binary Theory using the proof format given in Subsection 1.0.1, and any laws listed in Section 11.3. Do not use the Completion Rule.

- (a) $a \wedge b \Rightarrow a \vee b$
- (b) $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \equiv (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$
- (c) $\neg a \Rightarrow (a \Rightarrow b)$
- (d) $a = (b \Rightarrow a) \equiv a \vee b$
- (e) $a = (a \Rightarrow b) \equiv a \wedge b$
- (f) $(a \Rightarrow c) \wedge (b \Rightarrow \neg c) \Rightarrow \neg(a \wedge b)$
- (g) $a \wedge \neg b \Rightarrow a \vee b$
- (h) $(a \Rightarrow b) \wedge (c \Rightarrow d) \wedge (a \vee c) \Rightarrow (b \vee d)$
- (i) $a \wedge \neg a \Rightarrow b$
- (j) $(a \Rightarrow b) \vee (b \Rightarrow a)$
- (k) $\neg(a \wedge \neg(a \vee b))$
- (l) $(\neg a \Rightarrow \neg b) \wedge (a \neq b) \vee (a \wedge c \Rightarrow b \wedge c)$
- (m) $(a \Rightarrow \neg a) \Rightarrow \neg a$
- (n) $(a \Rightarrow b) \wedge (\neg a \Rightarrow b) \equiv b$
- (o) $(a \Rightarrow b) \Rightarrow a \equiv a$
- (p) $a = b \vee a = c \vee b = c$
- (q) $a \wedge b \vee a \wedge \neg b \equiv a$
- (r) $a \Rightarrow (b \Rightarrow a)$
- (s) $a \Rightarrow a \wedge b \equiv a \Rightarrow b \equiv a \vee b \Rightarrow b$
- (t) $(a \Rightarrow a \wedge b) \vee (b \Rightarrow a \wedge b)$
- (u) $(a \Rightarrow (p = x)) \wedge (\neg a \Rightarrow p) \equiv p = (x \vee \neg a)$
- (v) $(a \Rightarrow b \Rightarrow \neg a) \vee (b \wedge c \Rightarrow a \wedge c)$
- (w) $a = (b \wedge c) \wedge d = (\neg b \wedge \neg c) \wedge e = ((a \vee d) = c) \Rightarrow e = b$

After trying the question, scroll down to the solution.

§(a)	$a \wedge b$	specialization
	$\Rightarrow a$	generalization
	$\Rightarrow a \vee b$	
§(b)	$(a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$	distribute
	$= (a \vee b \vee c) \wedge (a \vee b \vee a) \wedge (a \vee c \vee c) \wedge (a \vee c \vee a) \wedge (b \vee b \vee c) \wedge (b \vee b \vee a) \wedge (b \vee c \vee c) \wedge (b \vee c \vee a)$	symmetry and idempotence
	$= \underline{(a \vee b \vee c)} \wedge \underline{(a \vee b)} \wedge (a \vee c) \wedge (b \vee c)$	absorption
	$= (a \vee b) \wedge (a \vee c) \wedge (b \vee c)$	symmetry
	$= (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$	
§(c)	$\neg a \Rightarrow (a \Rightarrow b)$	portation
	$= \neg a \wedge a \Rightarrow b$	noncontradiction
	$= \perp \Rightarrow b$	base
	$= \top$	
§(d)	$\underline{(a = (b \Rightarrow a))} = a \vee b$	symmetry of =
	$= ((b \Rightarrow a) = a) = a \vee b$	associativity of =
	$= ((b \Rightarrow a) = \underline{a = (a \vee b)})$	symmetry of = and \vee
	$= ((b \Rightarrow a) = (b \vee a) = a)$	inclusion
	$= \top$	
§(e)	$\underline{(a = (a \Rightarrow b))} = a \wedge b$	symmetry of =
	$= ((a \Rightarrow b) = a) = a \wedge b$	associativity of =
	$= ((a \Rightarrow b) = \underline{a = (a \wedge b)})$	symmetry of =
	$= ((a \Rightarrow b) = (a \wedge b) = a)$	inclusion
	$= \top$	
§(f)	$(a \Rightarrow c) \wedge (b \Rightarrow \neg c)$	law of conflation
	$\Rightarrow a \wedge b \Rightarrow c \wedge \neg c$	contrapositive law
	$= \neg(c \wedge \neg c) \Rightarrow \neg(a \wedge b)$	antecedent is law of noncontradiction
	$= \top \Rightarrow \neg(a \wedge b)$	identity for \Rightarrow
	$= \neg(a \wedge b)$	
§(g)	$a \wedge \neg b$	specialization
	$\Rightarrow a$	generalization
	$\Rightarrow a \vee b$	
§(h)	$(a \Rightarrow b) \wedge (c \Rightarrow d) \wedge (a \vee c) \Rightarrow (b \vee d)$	portation
	$= (a \Rightarrow b) \wedge (c \Rightarrow d) \Rightarrow (a \vee c \Rightarrow b \vee d)$	conflation
	$= \top$	
§(i)	$a \wedge \neg a \Rightarrow b$	noncontradiction
	$= \perp \Rightarrow b$	base
	$= \top$	
§(j)	$\underline{(a \Rightarrow b)} \vee \underline{(b \Rightarrow a)}$	inclusion, twice
	$= \neg a \vee b \vee \neg b \vee a$	symmetry of \vee
	$= \underline{a \vee \neg a} \vee \underline{b \vee \neg b}$	excluded middle, twice
	$= \top \vee \top$	idempotence of \vee , or base law
	$= \top$	

§(k) see book Subsection 1.0.2

$$\begin{aligned}
 \S(l) \quad & (\neg a \Rightarrow \neg b) \wedge (a \neq b) \vee (a \wedge c \Rightarrow b \wedge c) && \text{law of exclusion} \\
 = & (\neg a \Rightarrow \neg b) \wedge (a = \neg b) \vee (a \wedge c \Rightarrow b \wedge c) && \text{use context } a = \neg b \text{ to replace } \neg b \text{ with } a \\
 = & (\neg a \Rightarrow a) \wedge (a = \neg b) \vee (a \wedge c \Rightarrow b \wedge c) && \text{indirect proof} \\
 = & a \wedge (a = \neg b) \vee (a \wedge c \Rightarrow b \wedge c) && \text{context to replace second } a \text{ by } \top, \text{ and identity} \\
 = & a \wedge \neg b \vee (a \wedge c \Rightarrow b \wedge c) && \text{duality and double negation} \\
 = & \neg(\neg a \vee b) \vee (a \wedge c \Rightarrow b \wedge c) && \text{inclusion} \\
 = & \neg(a \Rightarrow b) \vee (a \wedge c \Rightarrow b \wedge c) \\
 & \text{context: use left disjunct to simplify the right disjunct; strengthen } b \text{ to } a \\
 \Leftarrow & \neg(a \Rightarrow b) \vee (a \wedge c \Rightarrow a \wedge c) && \text{reflexivity and base} \\
 = & \top
 \end{aligned}$$

Here's another solution.

$$\begin{aligned}
 & (\neg a \Rightarrow \neg b) \wedge (a \neq b) \vee (a \wedge c \Rightarrow b \wedge c) && \text{contrapositive, monotonicity} \\
 \Leftarrow & (b \Rightarrow a) \wedge (a \neq b) \vee (a \Rightarrow b) && \text{distributivity} \\
 = & ((b \Rightarrow a) \vee (a \Rightarrow b)) \wedge ((a \neq b) \vee (a \Rightarrow b)) && \text{material implication twice} \\
 = & (\neg b \vee a \vee \neg a \vee b) \wedge ((a \neq b) \vee (a \Rightarrow b)) && \text{symmetry and associativity} \\
 = & (a \vee \neg a \vee b \vee \neg b) \wedge ((a \neq b) \vee (a \Rightarrow b)) && \text{excluded middle twice, base or idempotent} \\
 = & (a \neq b) \vee (a \Rightarrow b) && \text{generic inequality} \\
 = & \neg(a = b) \vee (a \Rightarrow b) && \text{material implication} \\
 = & (a = b) \Rightarrow (a \Rightarrow b) && \text{antisymmetry (double implication)} \\
 = & (a \Rightarrow b) \wedge (b \Rightarrow a) \Rightarrow (a \Rightarrow b) && \text{specialization} \\
 = & \top
 \end{aligned}$$

$$\begin{aligned}
 \S(m) \quad & (a \Rightarrow \neg a) \Rightarrow \neg a && \text{material implication} \\
 = & (\neg a \vee \neg a) \Rightarrow \neg a && \text{idempotent} \\
 = & \neg a \Rightarrow \neg a && \text{reflexive} \\
 = & \top
 \end{aligned}$$

$$\begin{aligned}
 \S(n) \quad & (a \Rightarrow b) \wedge (\neg a \Rightarrow b) && \text{antidistributive law} \\
 = & a \vee \neg a \Rightarrow b && \text{antecedent is law of excluded middle} \\
 = & \top \Rightarrow b && \text{identity for } \Rightarrow \\
 = & b
 \end{aligned}$$

Here's another solution.

$$\begin{aligned}
 & (a \Rightarrow b) \wedge (\neg a \Rightarrow b) && \text{case analysis} \\
 = & \text{if } a \text{ then } b \text{ else } b \text{ fi} && \text{generic case idempotent} \\
 = & b
 \end{aligned}$$

$$\begin{aligned}
 \S(o) \quad & (a \Rightarrow b) \Rightarrow a && \text{context: use main consequent to simplify antecedent} \\
 = & (\perp \Rightarrow b) \Rightarrow a && \text{base} \\
 = & \top \Rightarrow a && \text{identity} \\
 = & a
 \end{aligned}$$

Here's another solution.

$$\begin{aligned}
 & (a \Rightarrow b) \Rightarrow a && \text{inclusion} \\
 = & (\neg a \vee b) \Rightarrow a && \text{inclusion} \\
 = & \neg(\neg a \vee b) \vee a && \text{duality} \\
 = & (\neg \neg a \wedge \neg b) \vee a && \text{double negation} \\
 = & (a \wedge \neg b) \vee a && \text{symmetry of } \vee \\
 = & a \vee (a \wedge \neg b) && \text{absorption} \\
 = & a
 \end{aligned}$$

$$\begin{aligned}
\text{\S(p)} \quad & a=b \vee a=c \vee b=c && \text{identity and reflexive laws for } = \\
= & a=b \vee a=c \vee \underline{b=((a=a)=c)} && \text{symmetry and associative laws for } = \\
= & (a=b \vee a=c) \vee (a=b)=(a=c) && \text{main } \vee \text{ distributes over } = \\
= & (a=b \vee a=c \vee a=b) = (a=b \vee a=c \vee a=c) && \text{symmetry, associativity, and idempotence of } \vee \text{ twice} \\
= & (a=b \vee a=c) = (a=b \vee a=c) && = \text{ is reflexive} \\
= & \top
\end{aligned}$$

Here's another solution.

$$\begin{aligned}
& a=b \vee a=c \vee b=c && \text{double negation} \\
= & \neg \neg(a=b) \vee a=c \vee b=c && \text{material implication} \\
= & \neg(a=b) \Rightarrow a=c \vee b=c && \text{inequality} \\
= & a \neq b \Rightarrow a=c \vee b=c && \text{exclusion} \\
= & a=\neg b \Rightarrow a=c \vee b=c && \text{context: use antecedent to modify consequent} \\
= & a=\neg b \Rightarrow \neg b=c \vee b=c && \text{exclusion} \\
= & a=\neg b \Rightarrow b \neq c \vee b=c && \text{inequality} \\
= & a=\neg b \Rightarrow \neg(b=c) \vee b=c && \text{symmetry, excluded middle} \\
= & a=\neg b \Rightarrow \top && \text{base} \\
= & \top
\end{aligned}$$

$$\begin{aligned}
\text{\S(q)} \quad & a \wedge b \vee a \wedge \neg b = a && \text{factor} \\
= & a \wedge (b \vee \neg b) = a && \text{excluded middle} \\
= & a \wedge \top = a && \text{symmetry} \\
= & \top \wedge a = a && \text{identity} \\
= & \top
\end{aligned}$$

$$\begin{aligned}
\text{\S(r)} \quad & a \Rightarrow (b \Rightarrow a) && \text{portation} \\
= & a \wedge b \Rightarrow a && \text{specialization} \\
= & \top
\end{aligned}$$

$$\begin{aligned}
\text{\S(s)} \quad & (a \Rightarrow a \wedge b = a \Rightarrow b = a \vee b \Rightarrow b) && \\
= & ((a \Rightarrow a) \wedge (a \Rightarrow b) = a \Rightarrow b = (a \Rightarrow b) \wedge (b \Rightarrow b)) && \text{distribute } \Rightarrow \text{ over } \wedge \text{ in first part; antidistribute } \Rightarrow \text{ over } \vee \text{ in last part} \\
= & (a \Rightarrow b = a \Rightarrow b = a \Rightarrow b) && \text{reflexivity of } \Rightarrow \text{ and identity of } \wedge \\
= & \top && \text{reflexivity of } =
\end{aligned}$$

$$\begin{aligned}
\text{\S(t)} \quad & (a \Rightarrow a \wedge b) \vee (b \Rightarrow a \wedge b) && \text{anti-distributive} \\
= & a \wedge b \Rightarrow a \wedge b && \text{reflexive} \\
= & \top
\end{aligned}$$

$$\begin{aligned}
\text{\S(u)} \quad & p=(x \vee \neg a) && \text{case idempotent law} \\
= & \text{if } a \text{ then } p=(x \vee \neg a) \text{ else } p=(x \vee \neg a) \text{ fi} && \text{context} \\
= & \text{if } a \text{ then } p=(x \vee \neg \top) \text{ else } p=(x \vee \neg \perp) \text{ fi} \\
= & \text{if } a \text{ then } p=(x \vee \perp) \text{ else } p=(x \vee \top) \text{ fi} \\
= & \text{if } a \text{ then } p=x \text{ else } p \text{ fi} && \text{case analysis} \\
= & (a \Rightarrow (p=x)) \wedge (\neg a \Rightarrow p)
\end{aligned}$$

Here's another solution.

$$\begin{aligned}
& (a \Rightarrow (p=x)) \wedge (\neg a \Rightarrow p) && \text{case analysis} \\
= & \text{if } a \text{ then } p=x \text{ else } p \text{ fi} && \text{identity} \\
= & \text{if } a \text{ then } p=x \text{ else } p=\top \text{ fi} && \text{case distributive} \\
= & p = \text{if } a \text{ then } x \text{ else } \top \text{ fi} && \text{one-case} \\
= & p = (a \Rightarrow x) && \text{material implication}
\end{aligned}$$

$$\begin{aligned} &= p = (\neg a \vee x) && \text{symmetry} \\ &= p = (x \vee \neg a) \end{aligned}$$

Here's another solution.

$$\begin{aligned} &(a \Rightarrow (p=x)) \wedge (\neg a \Rightarrow p) = p=(x \vee \neg a) && \text{case idempotent} \\ &= \mathbf{if\ } a \mathbf{\ then\ } (a \Rightarrow (p=x)) \wedge (\neg a \Rightarrow p) = p=(x \vee \neg a) && \text{context} \\ &\quad \mathbf{else\ } (a \Rightarrow (p=x)) \wedge (\neg a \Rightarrow p) = p=(x \vee \neg a) \mathbf{\ fi} && \text{context} \\ &= \mathbf{if\ } a \mathbf{\ then\ } (\top \Rightarrow (p=x)) \wedge (\neg \top \Rightarrow p) = p=(x \vee \neg \top) && \text{identity, base, identity} \\ &\quad \mathbf{else\ } (\perp \Rightarrow (p=x)) \wedge (\neg \perp \Rightarrow p) = p=(x \vee \neg \perp) \mathbf{\ fi} && \text{base, identity, base} \\ &= \mathbf{if\ } a \mathbf{\ then\ } (p=x) \wedge \top = p=x && \text{identity, reflexive} \\ &\quad \mathbf{else\ } \top \wedge p = p=\top \mathbf{\ fi} && \text{identity, reflexive} \\ &= \mathbf{if\ } a \mathbf{\ then\ } \top \mathbf{\ else\ } \top \mathbf{\ fi} && \text{case idempotent} \\ &= \top \end{aligned}$$

$$\begin{aligned} \S(v) &(a \Rightarrow b \Rightarrow \neg a) \vee (b \wedge c \Rightarrow a \wedge c) && \text{continuing implication} \\ &= (a \Rightarrow b) \wedge (b \Rightarrow \neg a) \vee (b \wedge c \Rightarrow a \wedge c) && \text{contrapositive and double negation} \\ &= (a \Rightarrow b) \wedge (a \Rightarrow \neg b) \vee (b \wedge c \Rightarrow a \wedge c) && \text{distributive} \\ &= (a \Rightarrow b \wedge \neg b) \vee (b \wedge c \Rightarrow a \wedge c) && \text{noncontradiction} \\ &= (a \Rightarrow \perp) \vee (b \wedge c \Rightarrow a \wedge c) && \text{double negation and indirect proof} \\ &= \neg a \vee (b \wedge c \Rightarrow a \wedge c) && \text{context} \\ &= \neg a \vee (b \wedge c \Rightarrow \top \wedge c) && \text{identity} \\ &= \neg a \vee (b \wedge c \Rightarrow c) && \text{specialization} \\ &= \neg a \vee \top && \text{base} \\ &= \top \end{aligned}$$

Here's another solution.

$$\begin{aligned} &(a \Rightarrow b \Rightarrow \neg a) \vee (b \wedge c \Rightarrow a \wedge c) && \text{continuing implication} \\ &= (a \Rightarrow b) \wedge (b \Rightarrow \neg a) \vee (b \wedge c \Rightarrow a \wedge c) && \text{monotonic} \\ \Leftarrow &(a \Rightarrow b) \wedge (a \Rightarrow \neg b) \vee (b \Rightarrow a) && \text{distributive} \\ &= ((a \Rightarrow b) \vee (b \Rightarrow a)) \wedge ((a \Rightarrow \neg b) \vee (b \Rightarrow a)) && \text{inclusion, twice} \\ &= (\neg a \vee b \vee \neg b \vee a) \wedge ((a \Rightarrow \neg b) \vee (b \Rightarrow a)) && \text{symmetry of } \vee \\ &= (a \vee \neg a \vee b \vee \neg b) \wedge ((a \Rightarrow \neg b) \vee (b \Rightarrow a)) && \text{excluded middle, twice} \\ &= (\top \vee \top) \wedge ((a \Rightarrow \neg b) \vee (b \Rightarrow a)) && \text{idempotence of } \vee, \text{ or base law} \\ &= \top \wedge ((a \Rightarrow \neg b) \vee (b \Rightarrow a)) && \text{identity} \\ &= (a \Rightarrow \neg b) \vee (b \Rightarrow a) && \text{contrapositive} \\ &= (a \Rightarrow \neg b) \vee (\neg a \Rightarrow \neg b) && \text{antidistributive} \\ &= a \wedge \neg a \Rightarrow \neg b && \text{noncontradiction} \\ &= \perp \Rightarrow \neg b && \text{base} \\ &= \top \end{aligned}$$

$$\begin{aligned} \S(w) &a=(b \wedge c) \wedge d=(\neg b \wedge \neg c) \wedge e=((a \vee d)=c) && \text{context: use first two conjuncts to modify last conjunct} \\ &= a=(b \wedge c) \wedge d=(\neg b \wedge \neg c) \wedge e=((b \wedge c \vee \neg b \wedge \neg c)=c) && \text{specialization} \\ \Rightarrow &e=((b \wedge c \vee \neg b \wedge \neg c)=c) && \text{equality and difference} \\ &= e=((b=c)=c) && \text{associativity} \\ &= (e=b)=(c=c) && \text{reflexivity, identity} \\ &= e=b \end{aligned}$$