

81 (arity) The arity of a function is the number of variables (parameters) it introduces, and the number of arguments it can be applied to. Write axioms to define αf (arity of f).

After trying the question, scroll down to the solution.

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As a first effort, we might try

$$\alpha x = 0 \quad \text{if } x \text{ is not a function}$$

$$\alpha(A \rightarrow B) = 1 + \alpha B$$

Unfortunately, $\text{null} \rightarrow \text{null} \rightarrow 3 = \text{null} \rightarrow 3$, so we would prove $2=1$. So we try again.

$$\alpha x = 0 \quad \text{if } x \text{ is not a function}$$

$$\alpha f = 1 + \alpha(f(\Box f)) \quad \text{if } f \text{ is a function}$$

Let's try an example.

$$\alpha \langle n: \text{nat} \cdot n=0 \models 0 \Rightarrow \langle m: \text{nat} \cdot m \rangle \rangle$$

$$= 1 + \alpha(f \text{ nat})$$

$$= 1 + \alpha(0, \langle m: \text{nat} \cdot m \rangle) \quad \text{Maybe we should let } \alpha \text{ distribute over bunch union.}$$

$$= 1 + (\alpha 0, \alpha \langle m: \text{nat} \cdot m \rangle)$$

$$= 1 + (0, 1 + \alpha(\langle m: \text{nat} \cdot m \rangle \text{ nat}))$$

$$= 1 + (0, 1 + \alpha \text{ nat})$$

$$= 1 + (0, 1 + 0)$$

$$= 1, 2$$

and that seems reasonable to me.