

- 9 Consider a fully bracketed expression containing only the symbols \top \perp $=$ \neq $()$ in any quantity and any syntactically acceptable order.
- (a) Show that all syntactically acceptable rearrangements are equivalent.
 - (b) Show that it is equivalent to any expression obtained from it by making an even number of the following substitutions: \top for \perp , \perp for \top , $=$ for \neq , \neq for $=$.

After trying the question, scroll down to the solution.

§ The proofs will be by induction over the structure of the expressions. Every fully parenthesized expression containing only the symbols $\top \perp = \neq ()$ has one of the following four forms: \top , \perp , $(a=b)$, $(a\neq b)$, where a and b are fully parenthesized expression containing only the symbols $\top \perp = \neq ()$.

(a) Show that all syntactically acceptable rearrangements are equivalent.

§ There are four alternatives. The first two alternatives are just a single symbol, so there are no rearrangements, so all zero rearrangements are equivalent. That's the base case. Now for the induction step.

Suppose the expression is $(a=b)$ for some expressions a and b . The ways of rearranging $(a=b)$ are:

- (a) rearrange a
- (b) rearrange b
- (c) change $(a=b)$ to $(b=a)$

First, consider (a). Make the inductive hypothesis that rearranging a results in an expression that is equivalent to a . Then any expression with subexpression a is equivalent to the same expression with subexpression a replaced by its rearrangement. (This is formalized as the generic law of transparency.) Similarly for (b). For (c), we have the generic law of symmetry of $=$. That completes the proof for expressions of the form $(a=b)$.

Finally, suppose the expression is $(a\neq b)$ for some expressions a and b . The ways of rearranging $(a\neq b)$ are:

- (a) rearrange a
- (b) rearrange b
- (c) change $(a\neq b)$ to $(b\neq a)$

First, consider (a). Make the inductive hypothesis that rearranging a results in an expression that is equivalent to a . Then any expression with subexpression a is equivalent to the same expression with subexpression a replaced by its rearrangement. (This is the transparency proof rule.) Similarly for (b). For (c), we have the generic law of symmetry of \neq .

That completes the proof.

(b) Show that it is equivalent to any expression obtained from it by making an even number of the following substitutions: \top for \perp , \perp for \top , $=$ for \neq , \neq for $=$.

§ Zero substitutions means the same expression, which is obviously equivalent. I will show that by making a single one of those substitutions, the expression is negated. Therefore two substitutions are a double negation, which is an equivalent expression. And so on for more substitutions.

If the expression is \top , the only substitution is \perp for \top , and \perp is the negation of \top .

If the expression is \perp , the only substitution is \top for \perp , and \top is the negation of \perp .

Suppose the expression is $(a=b)$ for some expressions a and b . The ways of making one substitution in $(a=b)$ are:

- (i) make one substitution in a

- (ii) make one substitution in b
- (iii) change $(a=b)$ to $(a\neq b)$

First, consider (i). Make the inductive hypothesis that one substitutions in a negates a , resulting in an expression equivalent to $(\neg a=b)$.

$$\begin{aligned} & (\neg a=b) && \text{exclusion} \\ = & (a\neq b) && \text{generic inequality} \\ = & \neg(a=b) \end{aligned}$$

so making one substitution in a negates $(a=b)$. Similarly for (ii). For (iii),

$$\begin{aligned} & (a\neq b) && \text{generic inequality} \\ = & \neg(a=b) \end{aligned}$$

That completes the proof for expressions of the form $(a=b)$. Finally, suppose the expression is $(a\neq b)$ for some expressions a and b . The ways of making one substitution in $(a\neq b)$ are:

- (i) make one substitution in a
- (ii) make one substitution in b
- (iii) change $(a\neq b)$ to $(a=b)$

First, consider (i). Make the inductive hypothesis that one substitutions in a negates a , resulting in an expression equivalent to $(\neg a\neq b)$.

$$\begin{aligned} & (\neg a\neq b) && \text{generic inequality} \\ = & \neg(\neg a=b) && \text{exclusion} \\ = & \neg(a\neq b) \end{aligned}$$

so making one substitution in a negates $(a\neq b)$. Similarly for (ii). For (iii),

$$\begin{aligned} & (a=b) && \text{double negation} \\ = & \neg\neg(a=b) && \text{generic inequality} \\ = & \neg(a\neq b) \end{aligned}$$

That completes the proof.