X3.0 (duality) Prove the duality laws on page 241:

$$\neg\forall v\cdot\ b \ = \ \exists v\cdot\ \neg b$$
$$\neg\exists v\cdot\ b \ = \ \forall v\cdot\ \neg b$$

from the quantifier axioms on page 240:

(0) $\forall v{:}\ null\cdot\ b \ = \ \top$
(1) $\forall v{:}\ x\cdot\ b \ = \ \langle v{:}\ x\cdot\ b\rangle\ x$
(2) $\forall v{:}\ A,B\cdot\ b \ = \ (\forall v{:}\ A\cdot\ b) \wedge (\forall v{:}\ B\cdot\ b)$
(3) $\forall v{:}\ (\S v{:}\ D\cdot\ b)\cdot\ c \ = \ \forall v{:}\ D\cdot\ b \Rightarrow c$

(4) $\exists v{:}\ null\cdot\ b \ = \ \bot$
(5) $\exists v{:}\ x\cdot\ b \ = \ \langle v{:}\ x\cdot\ b\rangle\ x$
(6) $\exists v{:}\ A,B\cdot\ b \ = \ (\exists v{:}\ A\cdot\ b) \vee (\exists v{:}\ B\cdot\ b)$
(7) $\exists v{:}\ (\S v{:}\ D\cdot\ b)\cdot\ c \ = \ \exists v{:}\ D\cdot\ b \wedge c$

After trying the question, scroll down to the solution.

§ I'll prove

$$\neg \forall v{:}\ D\cdot\ b \ = \ \exists v{:}\ D\cdot\ \neg b$$

by structural induction on the structure of the domain D. The ways of forming a domain are: $D = null$, $D = x$ (an element), $D = A,B$ (a union), and $D = \S v{:}\ E\cdot\ c$ (a comprehension). A domain formed as an intersection can also be formed as a comprehension: $A`B = \S v{:}\ A\cdot\ v{:}\ B$.

	$\neg \forall v{:}\ null\cdot\ b$	use (0)
$=$	$\neg \top$	
$=$	$\bot$	use (4)
$=$	$\exists v{:}\ null\cdot\ \neg b$	

	$\neg \forall v{:}\ x\cdot\ b$	use (1)
$=$	$\neg\ \langle v{:}\ x\cdot\ b\rangle\ x$	this line is: $\neg($ b but replace v by x $)$
$=$	$\langle v{:}\ x\cdot\ \neg b\rangle\ x$	this line is: $(\neg b$ but replace v by $x)$; now use (5)
$=$	$\exists v{:}\ x\cdot\ \neg b$	

	$\neg \forall v{:}\ A,B\cdot\ b$	use (2)
$=$	$\neg((\forall v{:}\ A\cdot\ b) \wedge (\forall v{:}\ B\cdot\ b))$	now apply the basic de Morgan
$=$	$\neg(\forall v{:}\ A\cdot\ b) \vee \neg(\forall v{:}\ B\cdot\ b)$	now a structural induction step, twice
$=$	$(\exists v{:}\ A\cdot\ \neg b) \vee (\exists v{:}\ B\cdot\ \neg b)$	use (6)
$=$	$\exists v{:}\ A,B\cdot\ \neg b$	

	$\neg \forall v{:}\ (\S v{:}\ E\cdot\ c)\cdot\ b$	use (3)
$=$	$\neg(\forall v{:}\ E\cdot\ c \Rightarrow b)$	now a structural induction step
$=$	$\exists v{:}\ E\cdot\ \neg(c \Rightarrow b)$	some binary algebra
$=$	$\exists v{:}\ E\cdot\ c \wedge \neg b$	use (7)
$=$	$\exists v{:}\ (\S v{:}\ E\cdot\ c)\cdot\ \neg b$	

And the other duality law

$$\neg \exists v{:}\ D\cdot\ b \ = \ \forall v{:}\ D\cdot\ \neg b$$

can be proven similarly. Or we can prove it from the one we just proved.

	$\neg \exists v{:}\ D\cdot\ b$	double negation
$=$	$\neg \exists v{:}\ D\cdot\ \neg\neg b$	use what we just proved
$=$	$\neg\neg \forall v{:}\ D\cdot\ \neg b$	double negation
$=$	$\forall v{:}\ D\cdot\ \neg b$	