X5.3 Given a standard 2-sided coin, probability $1/2$ for each side, build a 3-sided die, probability $1/3$ for each side. (I have no idea what a 3-sided die would look like, but we can write a probabilistic program for it.) How many times do you have to flip a coin in order to roll your 3-sided die?

After trying the question, scroll down to the solution.

§ We can build a 3-sided die as follows:

$$\textbf{if } 1/3 \textbf{ then } d:= 0 \textbf{ else if } 1/2 \textbf{ then } d:= 1 \textbf{ else } d:= 2 \textbf{ fi fi}$$

But this solution requires that we already have a way to get probability $1/3$. We want to build $1/3$ from $1/2$. Define specification S as follows:

$$S = ((d'=0)+(d'=1)+(d'=2))/3$$

Here's the idea. If you flip a coin twice, there are four possible outcomes, which is one too many. So pick one of the four, and keep flipping twice until the outcome is one of the other three. The program is

$$S = \textbf{if } 1/2 \textbf{ then if } 1/2 \textbf{ then } S \textbf{ else } d:= 0 \textbf{ fi else if } 1/2 \textbf{ then } d:= 1 \textbf{ else } d:= 2 \textbf{ fi fi}$$

To prove it, I start with the right side.

$\textbf{if } 1/2 \textbf{ then if } 1/2 \textbf{ then } S \textbf{ else } d:= 0 \textbf{ fi else if } 1/2 \textbf{ then } d:= 1 \textbf{ else } d:= 2 \textbf{ fi fi}$

replace S with its definition, and replace **if**s and assignments

$= \quad 1/2 \times (1/2 \times ((d'=0)+(d'=1)+(d'=2))/3 \ + \ 1/2 \times (d'=0))$
$\quad + \ 1/2 \times (1/2 \times (d'=1) \ + \ 1/2 \times (d'=2))$

$= \quad (1/2 \times 1/2 \times 1/3 \ + \ 1/2 \times 1/2)\times(d'=0)$
$\quad + \ (1/2 \times 1/2 \times 1/3 \ + \ 1/2 \times 1/2)\times(d'=1)$
$\quad + \ (1/2 \times 1/2 \times 1/3 \ + \ 1/2 \times 1/2)\times(d'=2)$

$= \quad (1/12 + 1/4)\times(d'=0) \ + \ (1/12 + 1/4)\times(d'=1) \ + \ (1/12 + 1/4)\times(d'=2)$

$= \quad 4/12 \times (d'=0) \ + \ 4/12 \times (d'=1) \ + \ 4/12 \times (d'=2)$

$= \quad ((d'=0)+(d'=1)+(d'=2))/3$

$= \quad S$

For the execution time, we can count flips by putting $t:= t+1$ before each flip.

$$S = t:= t+1.\ \textbf{if } 1/2 \textbf{ then } t:= t+1.\ \textbf{if } 1/2 \textbf{ then } S \textbf{ else } d:= 0 \textbf{ fi}$$
$$\textbf{else } t:= t+1.\ \textbf{if } 1/2 \textbf{ then } d:= 1 \textbf{ else } d:= 2 \textbf{ fi fi}$$

If we focus on t and ignore d , we can simplify this equation to

S

$= \quad t:= t+1.\ \textbf{if } 1/2 \textbf{ then } t:= t+1.\ \textbf{if } 1/2 \textbf{ then } S \textbf{ else } ok \textbf{ fi}$
$\qquad \textbf{else } t:= t+1.\ \textbf{if } 1/2 \textbf{ then } ok \textbf{ else } ok \textbf{ fi fi}$

Factor out $t:= t+1$, and simplify $\textbf{if } 1/2 \textbf{ then } ok \textbf{ else } ok \textbf{ fi}$ to ok .

$= \quad t:= t+2.\ \textbf{if } 1/2 \textbf{ then if } 1/2 \textbf{ then } S \textbf{ else } ok \textbf{ fi else } ok \textbf{ fi}$ — replace **if** and ok

$= \quad t:= t+2.\ 1/2\times(1/2\times S + 1/2\times(t'=t)) + 1/2\times(t'=t)$

$= \quad t:= t+2.\ S/4 + (t'=t)\times 3/4$ — substitution law

$= \quad (t:= t+2.\ S)/4 + (t'=t+2)\times 3/4$

$= \quad (t'=t+2)\times(3/4)$
$\quad + \ (t'=t+4)\times(1/4)\times(3/4)$
$\quad + \ (t'=t+6)\times(1/4)\times(1/4)\times(3/4)$
$\quad + \ (t'=t+8)\times(1/4)\times(1/4)\times(1/4)\times(3/4)$
$\quad + \ (t'=t+10)\times(1/4)\times(1/4)\times(1/4)\times(1/4)\times(3/4)$
$\quad + \ ...$

$= \quad (even\ (t'-t))\times(t'\ge t+2)\times 3/2^{t'-t}$

That is the distribution of the number of flips.

Proof:

$(t:= t+2.\ S)/4 + (t'=t+2)\times 3/4$ — replace S

$= \quad (t:= t+2.\ (even\ (t'-t))\times(t'\ge t+2)\times 3/2^{t'-t})/4 + (t'=t+2)\times 3/4$ — substitution law

$=$ $(even\ (t'-t-2))\times(t'\geq t+4)\times 3/2^{t'-t-2}/4 + (t'=t+2)\times 3/4$ simplify *even* and divide by 4

$=$ $(even\ (t'-t))\times(t'\geq t+4)\times 3/2^{t'-t} + (t'=t+2)\times 3/4$ combine

$=$ $(even\ (t'-t))\times(t'\geq t+2)\times 3/2^{t'-t}$

$=$ S

The average value of t' is

$S.\ t$

$=$ $(even\ (t'-t))\times(t'\geq t+2)\times 3/2^{t'-t}.\ t$

$=$ $\Sigma t''\cdot\ (even\ (t''-t))\times(t''\geq t+2)\times 3/2^{t''-t}\times t''$

$=$ $3/4\times(t+2) + 3/16\times(t+4) + 3/64\times(t+6) + 3/256\times(t+8) + ...$ divide into two sums

$=$ $3/4\times t + 3/16\times t + 3/64\times t + 3/256\times t + ...$ factor out t

$+\ 3/4\times 2 + 3/16\times 4 + 3/64\times 6 + 3/256\times 8 + ...$

$=$ $(3/4 + 3/16 + 3/64 + 3/256 + ...)\times t + (3/4\times 2 + 3/16\times 4 + 3/64\times 6 + 3/256\times 8 + ...)$

$(3/4 + 3/16 + 3/64 + 3/256 + ...) = (3/4)/(1 - 1/4) = 1$

Let $x\ =\ 3/4\times 2 + 3/16\times 4 + 3/64\times 6 + 3/256\times 8 + ...$

Then $1/4\times x\ =\ 3/16\times 2 + 3/64\times 4 + 3/256\times 6 + ...$

So $3/4\times x\ =\ 3/4\times 2 + 3/16\times 2 + 3/64\times 2 + 3/256\times 2 + ...\ =\ (3/4\times 2)/(1 - 1/4)\ =\ 2$

Therefore $x\ =\ 8/3$

And $S.\ t\ =\ t + 8/3$

On average, there are $2 + 2/3$ coin flips.