X7.0 Let a, c, and x be natural variables. Variables a and c are implementer's variables, and x is a user's variable for the operations

$start = a := 1.\ c := 0$

$double = a := a \times 2.\ c := c + 1$

$ask = x := c$

Operation *start* starts variable a at 1. Then repeated use of operation *double* doubles it some number of times. Variable c counts how many times a is doubled. Operation *ask* asks how many times a has been doubled since the last *start* operation. Reimplement this theory replacing the old implementer's variable a with nothing.

(a) What is the data transformer? Prove it is a data transformer.

(b) Using your data transformer, transform *double*.

After trying the question, scroll down to the solution.

(a) What is the data transformer? Prove it is a data transformer.

§ One data transformer is

$\top$

We need to prove $\forall new \cdot \exists old \cdot D$. There are no new implementer's variables, so there is no $\forall$ quantifier. We are replacing only a , so we prove

$\exists a \cdot \top$ idempotent

$=$ $\top$

Another data transformer is

$2^c = a$

We prove

$\exists a \cdot 2^c = a$ identity

$=$ $\exists a \cdot 2^c = a \ \wedge \ \top$ one-point on a

$=$ $\top$

(b) Using your data transformer, transform *double* .

§ Using data transformer $\top$,

$\forall a \cdot \top \Rightarrow \exists a' \cdot \top \wedge double$

$=$ $\forall a \cdot \exists a' \cdot double$

$=$ $\forall a \cdot \exists a' \cdot (a := a \times 2 .\ c := c+1)$

$=$ $\forall a \cdot \exists a' \cdot a' = a \times 2 \ \wedge \ c' = c+1 \ \wedge \ x' = x$ one-point on a'

$=$ $\forall a \cdot c' = c+1 \ \wedge \ x' = x$ unused a

$=$ $c' = c+1 \ \wedge \ x' = x$ assignment

$=$ $c := c+1$

Using data transformer $2^c = a$,

$\forall a \cdot 2^c = a \ \Rightarrow \ \exists a' \cdot 2^{c'} = a' \wedge double$

$=$ $\forall a \cdot 2^c = a \ \Rightarrow \ \exists a' \cdot 2^{c'} = a' \wedge (a := a \times 2 .\ c := c+1)$

$=$ $\forall a \cdot 2^c = a \ \Rightarrow \ \exists a' \cdot 2^{c'} = a' \ \wedge \ a' = a \times 2 \ \wedge \ c' = c+1 \ \wedge \ x' = x$ one-point on a'

$=$ $\forall a \cdot 2^c = a \ \Rightarrow \ 2^{c'} = a \times 2 \ \wedge \ c' = c+1 \ \wedge \ x' = x$ one-point on a

$=$ $2^{c'} = 2^c \times 2 \ \wedge \ c' = c+1 \ \wedge \ x' = x$ context

$=$ $2^{c+1} = 2^c \times 2 \ \wedge \ c' = c+1 \ \wedge \ x' = x$ arithmetic

$=$ $c' = c+1 \ \wedge \ x' = x$ assignment

$=$ $c := c+1$