[1] [face] Hello. I'm Eric Hehner, from the computer science department, of the University of Toronto. This video is about number representation. Our normal, everyday number representation that we all use is base ten, which means it has ten different digits. Why base ten? Everyone with ten fingers gives the same answer: they hold up their ten fingers. And I'm sure they're right; that is why we use base ten. In fact, the word digit means finger. In base ten, the digits are zero, one, two, three, four, five, six, seven, eight, and nine. In base two they are zero and one. In base eight they are zero, one, two, three, four, five, six and seven. In base n, there are n digits, and they go from zero up to one less than the base. So what digits can I show with my hands? I can show zero, one, two, three, four, five, six, seven, eight, nine, and ten. That's eleven different digits. It's base eleven. Our hands say we should be using base eleven. I'll let you think about that. I'll use base ten in this video, the same as everyone else, because it's the standard, and it's familiar.

[2] There's the number one thousand two hundred and thirty four. The way we represent numbers today, as a sequence of digits, was invented in India about eighteen hundred years ago. It's a brilliant idea, one of the best ideas people have ever had. Since then, there have been three additions to the notation. [3] First, there's the point. That's twelve point three four. [4] Next, there's the rep, which is a raised dot. That's one two rep three four. And [5] finally there's the quote. That's one two quote three four. It's these three notations, the point, the rep, and the quote, that this video is about. So let's start with [6] the point. If we are using [7] decimal digits, we call it a decimal point. If we are using [8] binary digits, then it's a binary point. And similarly for other bases. [9] In some parts of the world, a comma is used instead, but it's the same idea. [10] When the point is one digit from the right end, it means division by ten. [11] Move it left one more digit, and it's division by a hundred. [12] Left again, and it's division by a thousand. [13] It can even be at the left end of the digits. And we can keep moving it left [14] because there are invisible zeros to the left of the visible digits. [15] The invisible zeros become visible so that the point can be moved leftward. The point can also be [16] at the right end, which has no effect on the number. We can say there's always a point, but when it's at the right end, we might not bother to write it; it's there but it's invisible. [17] We can move it further right because there are invisible zeros to its right. [18] And again, and we could keep going. With just a digit sequence, we can represent the natural numbers, zero, one, two, three, and so on. With a digit sequence and a point, [19] we can represent those nonnegative rational numbers that are the result of dividing a natural number by a power of the base. That's a lot more numbers. We were all taught what a point is and how to use it in school. But some people don't learn. If you write [20] twelve point three dollars, or twelve point three four five dollars, some people are confused because they want to see two digits after the point. I've seen [21] this marked as the price of something that costs ten cents, but that isn't ten cents; it's a tenth of a cent. And I once saw [22] this as the price of gas; it doesn't make any sense.

[23] All right, let's move on to the rep. [24] I was taught in school to put a line over some digits to mean that they are repeated forever. But a line above digits is not convenient for typing, so instead, [25] a rep is used for that same purpose. It means that the digits that come after the rep are repeated forever. We won't be using [26] a line over top. -- The rep [27] doesn't have to be to the right of the point. This rep tells us the digits are one two three four five three four five three four five, and so on forever, with the point coming after the first three. [28] A point and a rep can be in the same position, and we use a colon for that. [29] A rep can be at the left end of the digits, meaning that all the digits are repeated. [30] A rep can be at the right end, in which case it's the invisible zeros to its right that are repeated. We can say that there's always a rep, but [31] when it's at the right end, we may not bother to write it. It's there, but invisible. [32] If there is a rep but no point, the point is invisibly at the right end of the visible digits.

A rep is produced by a division operation. For example, [33] we'll divide six hundred and eleven by four hundred and ninety five. It looks like four ninety five goes into six eleven [34] one time, so we [35] subtract one times four ninety five from six eleven and get one hundred and sixteen. Now, to the right of six eleven there's [36] an invisible point and an invisible zero, and we need to make them visible so we can bring down the zero. Now I guess that four ninety five goes into eleven sixty [37] twice. This guessing is the tough part. If you guess too big, you won't be able to do the subtraction. If you guess too small, the difference after the subtraction will be too big. [38] We subtract two times four ninety five from eleven sixty, and get one seventy. Now [39] we make another zero visible and bring it down. Now I guess [40] three for the next digit of the answer, and [41] subtract three times four ninety five, and get two fifteen. [42] We make another zero visible, and bring it down. This time I guess [43] four, so I subtract [44] four times four ninety five and get one seventy. [45] I make one more zero visible and bring it down and I see that we [46] had this number before, two subtractions ago, so I put [47] a rep two digits back, and that's the answer. One point two rep three four.

[48] The last notation is the quote. A quote is just like a rep but in the opposite direction. [49] For example, one two quote three four says that the digits one two are repeated forever to the left. A quote can come between digits, [50] or at the right end, which means all the digits are repeated to the left, [51] or at the left end, [52] which means that invisible zeros are repeated to the left. [53] We can say there's always a quote, but when it's at the left end, we may not bother to write it. [54] It's there but invisible. We can have a [55] quote and a point, with the quote first, [56] or with the point first, or [57] with the quote and point in the same place, and we use an exclamation mark for that. Now [58] here's an example with a quote, a point, and a rep. The quote says that the one to its left is repeated forever to the left, and the rep says that the four to its right is repeated forever to the right. And [59] here's another example with all three marks. The rep says that the two three four to its right are repeated forever to the right, and the quote says that the one two three to its left are repeated forever to the left. And finally, [60] here's an example with all three marks in the same position, and we use a triplon for that. There's complete freedom in the positions of the three marks. But what does it all mean?

[61] Let's start with negation. What do we get when we negate one thousand two hundred and thirty four? Well first, [62] I need to put in the invisible quote and one invisible zero. The rule for negation is: [63] complement each digit and then add one. Complementing means change zero to nine, one to eight, two to seven, three to six, four to five, and so on up to changing nine to zero. So that's nine quote eight seven six five, plus one. Adding, [64] we get nine quote eight seven six six. That's the negation of twelve thirty four. [65] That's the radix complement rule, which might be familiar to you as two's complement in base two. Now, just to check [66] let's negate nine quote eight seven six six. [67] First, complement each digit, and then add one. And we get [68] back what we had before. As another check, [69] let's add these two numbers. [70] Starting at the right end, six plus four is ten, so [71] put down the zero and carry the one. I'm putting carries in the middle, but you can still see the number ten. Now [72] we have to add six plus one plus three, and we get [73] ten again. Now add [74] seven plus one plus two, and get [75] ten again. Now add [76] eight plus one plus one, and get [77] ten again. Now add [78] nine plus one plus zero, and get [79] ten again. All the rest of the columns are exactly the same, so we can put in the [80] quote mark, and we have the answer: zero.

In that example, we were adding a negative number and a positive number, and all we did is just add. Now [81] let's add two negative numbers. All the negative integers start with nine quote. Nine quote is minus one, and nine quote eight is minus two. We start at the [82] right side. Nine plus eight is [83] seventeen, so put down the seven and carry the one.

Next [84] we have to add a nine and a one and another nine. The nine in the top row is there because nine quote means nine is repeated to the left. Maybe I should have made it visible, but I didn't. [85] So that's nineteen. Put down the nine and carry the one. Now it's going to be nine plus one plus nine again, and that keeps repeating, so we [86] put the quote in the answer and we're done. Nine quote seven is [87] minus three. To add two negative numbers, we just add. Here's [88] one more addition example. We're adding minus four plus six. [89] We start at the right, adding six plus six, and get [90] twelve. Now we add [91] nine plus one plus zero, and get [92] ten. From here on, it's always going to be nine plus one plus zero, [93] so we insert the quote, and we're done. [94] Minus four plus six is two.

Let's move on to [95] subtraction. Here's the way I was taught. You can't take six from four, so you borrow. But you can't borrow from zero, [96] so you borrow from the seven. That makes it a six, and it makes the zeros into nines, and it makes the four into fourteen. So now [97] fourteen minus six is eight, [98] nine minus two is seven, [99] nine minus five is four, [100] six minus three is three, [101] two minus zero is two, and we're done. That's a terrible way to subtract because borrowing has to reach arbitrarily far to the left, and it's not how calculators and computers subtract. [102] Here's a better way. You can't take six from four, so you [103] carry one to the next column. [104] There's the fourteen that we subtract six from, and get [105] eight. In the [106] next column, you add the one and two together, so we're subtracting three from zero. We have to [107] carry again, and there's the ten from which we subtract three, and get [108] seven. In the [109] next column, we're subtracting six from zero. So we [110] carry, and we subtract six from ten, and get [111] four. In the [112] next column, we're subtracting four from seven, and get [113] three. Then [114] zero from two is [115] two, and we're done. Same answer as before. That's how calculators and computers do it, and it's essential to subtract that way for numbers with quotes in them.

[116] Now let's subtract twelve thirty four from zero. This time I made the quote and some zeros visible, but I could have left them invisible. Starting at the right, four from zero, so that's [117] four from ten, which is [118] six. Now three pus one is four subtracted from zero again, so that's [119] four from ten, which is [120] six. Three from [121] ten is [122] seven. Two from [123] ten is [124] eight. One from [125] ten is [126] nine. Again it's going to be one from ten, and so on, so put in the [127] quote and we're done. Same answer as on the first line.

Ok, we've [128] added numbers, and we've [129] subtracted numbers with quote marks in them. No matter whether x and y are positive or negative, if it says add, we add, and if it says subtract, we subtract. Now [130] let's multiply. As an example, [131] multiply minus two, which is nine quote eight, times one hundred and twenty three. Starting at the right, three eights are twenty four, so [132] put down the four, and remember the two. Next, three nines are twenty seven, plus the two from before makes twenty nine, so put down [133] the nine and remember the two. But we're not done with the three yet because there's [134] another nine right here. So three nines is twenty seven, plus two makes twenty nine, so put down the nine and remember the two. This is repeated forever, so put in the [135] quote mark in the product. Anyway, three times minus two is minus six, which is nine quote four. Now we have to multiply times two, and move the answer over one place. I'll skip the details, and [136] just put the result. Two times minus two is minus four, which is nine quote six. And [137] one times nine quote eight is nine quote eight. Now we add. On the right, [138] four, then nine plus six is fifteen, so [139] put down the five and carry one. Now we have nine plus nine plus eight plus the carry, so that's [140] twenty seven. Next there's nine plus nine plus nine plus the carry two, that's [141] twenty nine, and from now on it's all the same, so put in the [142] quote, and we're done. And that's [143] minus two forty six. So multiplying works the same way as usual, no matter whether the operands are positive or negative. But a calculator or computer does not add up three or more numbers in one addition. It adds two numbers at a time. So let's do a multiplication the way a calculator or computer works, and just to change it a bit, [144] let's reverse the multiplier and multiplicand. First we multiply one twenty three times eight, and get [145] nine eighty four. The four is an answer digit, so we [146] put it up here. Now we multiply one twenty three times nine and get [147] eleven oh seven, and we [148] add the two numbers. The five is an answer digit, so [149] put it up at the top. Now multiply one twenty three times nine again, and get [150] eleven oh seven again, and [151] add two numbers. The seven is an answer digit, so [152] put it up at the top. Multiply one twenty three times nine again, and get [153] eleven oh seven again, and [154] add two numbers. The nine is an answer digit, so [155] put it up at the top. Now notice [156] that we have the same number we had one digit ago, so [157] put the quote one digit back, and we're done. Same answer as before.

[158] The division we did earlier produces digits from left to right, with a rep somewhere. Let's call that rep division. [159] Quote division produces digits from right to left with a quote somewhere. [160] Here's an example. Divide one ninety one by thirty three. Look at the rightmost digits of the two numbers, and ask: what times three ends in one? The answer is [161] seven, so that's the rightmost digit of the answer. Now multiply seven times thirty three and [162] get two thirty one, which we subtract, and get [163] nine quote six zero. We chose the answer digit, seven, so this difference will end in zero, so [164] we don't bother to write the zero. Now, the rightmost digit of this number is six, and the rightmost digit of the divisor is still three, so what times three ends in six? The answer is [165] two, so that's the next digit of the answer. Two times thirty three is [166] sixty six, which we subtract, and get [167] nine quote three zero, but [168] delete the zero. Now, what times three ends in three? It's [169] one. And one times thirty three is [170] thirty three, which we subtract and get [171] nine quote six zero, but I didn't write the zero. This is the same number we got [172] two subtractions ago, so [173] put in the quote two digits back, and we're done. When we did [174] rep division, we had to guess the next digit to make it as large as possible without making the multiplication so big that it can't be subtracted. With quote division, there's no guessing. There's exactly one digit that, when multiplied by [175] three, gives the current final digit. And the same would be true if the divisor ended in one or three or seven or nine. But not if it ended in zero, so let's try [176] this division. There isn't any digit times zero ends in one. But we can solve this problem just by [177] dividing both operands by ten. That puts a point in the dividend, and gets rid of the zero in the divisor. So now we [178] can do the division, and the answer has a point in it. What happens [179] if the divisor ends in a five? We just [180] multiply both operands times two, and then [181] get rid of the zero the same as before, and [182] the answer has a point in it. What happens [183] if the divisor ends in a two? We just [184] multiply both operands times five, and then [185] get rid of the zero the same as before, and the answer has a point in it. One more example. [186] One ninety one by six. The problem is that six has a factor of two in it, so we [187] multiply the operands times five, [188] get rid of the zero, and [189] divide. [190] In base ten we have to get rid of factors two and five from the divisor. And that's a big annoyance. [191] If we were using a prime base, we wouldn't have any of this bother. So base eleven, which we should be using because of our ten fingers, is looking pretty good. Base two is also a prime number, and that's what computers use, so they don't have the problem.

Now I want to show you how to tell if a number is greater than zero or less than zero. For example, [192] one two quote three four. Just [193] look at the first digit, and the first digit after the quote. If the first digit is bigger, the number is negative. If the first digit after the quote is bigger, the number is positive. Since three is more than one, [194] this number is positive. In [195] this example, [196] five is more than three, so [197] this number is negative. And if they're [198] equal, as in this example, then you need to [199] roll the repeating part to the right. That's the same digit sequence, and now we see [200] that the number is negative. If [201] there's nothing to the left of the quote, there's an invisible zero there. And if [202] there's nothing to the right of the quote, there's an invisible zero there.

[203] Ok, let's see what we've got. [204] A sequence of digits gives us the natural numbers, zero, one, two, and so on. [205] Digits plus a point gives us those nonnegative rationals that are a natural number divided by a natural power of the base. And if we [206] add a rep, we get all nonnegative rational numbers. But there's a problem. We can't do any arithmetic with them because addition starts at the right end, and there's no right end. Same for subtraction and multiplication. Division starts at the left, but it uses subtractions, and they start at the right end. So in general, except for a few special cases, like if it's just zeros that repeat, a rep makes arithmetic impossible. [207] With just a quote, we get those rationals that are not the result of dividing an integer by a power of the base. So we get one third, but we don't get one half. [208] With a quote and a point we get all the rationals. That's more than we get with a rep and a point. [209] With a quote, a rep, and a point, we still just get the rationals, but the rep spoils our ability to do arithmetic. But don't throw the rep away, because it's the best for making approximations, and I think that's really important.

[210] [211] Here are all the usual operations on numbers that have a number result. [212] I'm going to use the word evaluation with the following meaning. If you have any number expression with any operators in it, you evaluate the expression by finding an equivalent expression that uses only digits, a point, a rep, and a quote. You get rid of the operators. [213] There are also expressions that have number operands and a binary result. Then evaluation means re-expressing it as either true or false. [214] Here is the first example: one twenty three plus forty five. We [215] do the addition, and get one sixty eight. The [216] next example has two operators in it: a subtraction and a negation, and there's more than one way to do this. One way is [217] to reverse the subtraction, which gives us another negation. [218] Now do the subtraction. And now the two negations [219] cancel each other, and we're done. [220] Here's a similar example. In this one, we can start by [221] doing the subtraction. Now I want to make an important point. The [222] minus sign that's in front of this five is the same minus sign that was in front of the subtraction, and the same as the minus signs in the [223] previous example. It's an operator, with an operand. It's not part of a number representation. The so-called sign and magnitude representation. People who have been told that it is the sign and magnitude representation often make the following mistake. [224] They think that minus x is negative. That's because they have been told that a minus sign in front means it's negative. But it could be negative, zero, or positive, depending on its operand. Minus five is not an answer. It's a question. What do you get when you apply the minus operator to the number 5? And it has an answer. It's [225] nine quote five. Minus five is unevaluated. Nine quote five is evaluated. In the same way that [226] one divided by three is unevaluated. One third is not an answer. We can evaluate it using rep division and get [227] point three repeated. Or we can use quote division and get [228] six quote seven. That's not to say there is anything wrong with unevaluated expressions. For some purposes they are more useful than evaluated expressions. Minus five and one third are easily understood. Just like [229] scientific notation. We can [230] evaluate it, but the unevaluated form saves us from counting zeros in large numbers. Also [231] in small numbers, scientific notation is usually more useful than [232] evaluating it. In the [233] next example, evaluation results in a number with twenty thousand digits in it, so it's completely impractical to evaluate it. [234] The next example is the square root of two. It cannot be evaluated completely because the digits are not ultimately repeating. But we can

get a very good approximation. In the [235] last example, the square root of minus one, we can't even get an approximation. It can't be evaluated.

[236] Now we look at unevaluation, which is the opposite of evaluation. It means getting rid of points and reps and quotes, replacing them with operators. [237] Getting rid of a point is easy. [238] You just divide by a power of the base. [239] Here's a point and a rep. [240] Let x be that number. The repeating part is two digits long, so [241] multiply x times a hundred, which just means moving the point right two places. The reason for doing this is to create a new number that has the same repeating digits in the same places. So [242] we can subtract and that [243] gets rid of all the repeating digits. So [244] ninety nine times x equals one twenty two point two. So x, the original number, [245] equals one twenty two point two over ninety nine. Now we just have to get rid of the point, so [246] we multiply both dividend and divisor times ten and we're done. [247]

The next example [248] has a point and a quote. Like the previous example, [249] let x be that number. The repeating part is two digits long, so [250] multiply x times a hundred, which just means moving the point right two places. The reason for doing this is the same as before: to create a new number that has the same repeating digits in the same places. So [251] we can subtract and that [252] gets rid of all the repeating digits. So [253] ninety nine times x equals two point one six six. So x, the original number, [254] equals two point one six six over ninety nine. Now we just have to get rid of the point, so [255] we multiply both dividend and divisor times a thousand and we're done. [256]

Here's [257] another example with a point and a quote. [258] Let x be that number. The repeating part is again two digits long, so [259] multiply x times a hundred. Now [260] subtract, and this time I'll do it in detail. [261] Starting at the right side, one from zero, so make it one from ten is [262] nine. In the [263] next column, that's three from zero. It's three, not two, because of the carry. So three from ten is [264] seven. In the [265] next column, it's four from one, so make it four from eleven, and that's [266] seven. [267] Next, five from twelve, which is [268] seven. [269] Next, four from thirteen, which is [270] nine. [271] Next, five from fourteen, which is [272] nine, and all the rest are going to be the same as the previous two, so put in the [273] quote, and we're done. [274] Ninety nine times x is nine quote seven seven seven point nine. So x, the original number, is [275] nine quote seven seven seven point nine over ninety nine. Getting rid of the point is easy [276] but we still have a quote. Nine quote means it's a negative integer, so we can just [277] negate it. [278] We could have done the steps of the unevaluation in a different order.

[279] Numbers have multiple representations. In these examples, I'm going to make all three marks visible. [280] Here's the first example. On the left, the rep says to repeat two three two three. So that can be shortened to just two three. [281] The same thing happens with the quote. [282] In this one, on the left it's point one two -- three two three two and so on. So the rep mark could have been one digit sooner. [283] And the same thing happens with the quote. When the first digit equals the first digit after the quote, you can shorten it. [284] You probably know that point nine repeated is equal to one. Well, whenever nines are repeated to the right, you can get rid of them by increasing the previous digit. [285] Here's the proof. Since it's one digit repeated, multiply times ten. Then subtract the original number. That gets rid of the repeated nines. Then divide by nine, and get your answer. [286] Here's one we've seen before. Two ways to evaluate one divided by three. [287] Leading zeros might be visible or invisible, and [288] the same for trailing zeros. [289] The next one is the number zero. That's both a leading and a trailing zero, so it can be invisible, leaving nothing but the three marks, quote, rep, and point. [290] This next one says that rep one two three quote point is equal to zero. So that needs some explanation. [291] The number means the sequence one two three stretching forever in both directions, with a point between a three and a one. Let's call this x. Since the repeating part is three digits, we [292] multiply

times a thousand. But [293] that's exactly the same number. Since x equals a thousand times x, therefore [294] x equals zero. It could have been any digit sequence preceded by a rep and followed by a quote, and the point could be anywhere, and it would be equal to zero.

[295] Now we look at how computation is done, and we start with unevaluated expressions of the form a over b and minus a over b. [296] Here is an example addition. First you notice that the left operand isn't negated, and the right operand is. So we [297] change it into a subtraction. The divisors are different, so we make them into the same divisor [298] with some multiplications, and then we [299] do those multiplications. Now we see that the subtraction's left operand is smaller than the right operand, so [300] we turn that around with a negation out front. [301] And then we do the subtraction. But we're not done. We have to [302] reduce the dividend and divisor by their greatest common divisor. And [303] how do we find that? To find the gcd of two numbers, [304] subtract the smaller number from the larger number [305]. Then again, [306] subtract the smaller number from the larger number [307]. And [308] again and again until the two numbers are equal, and [309] that's the answer. So back to the main calculation, [310] divide dividend and divisor by two, [311] and that's the final answer. That's a whole lot of work just to add two numbers in this unevaluated form.

[312] What does a calculator or computer do? [313] Same example. It starts by doing the divisions [314], using rep division, but it approximates the answers instead of using a rep mark. Now, as before, the left operand isn't negated and the right operand is. So it [315] changes into a subtraction. Now we compare the operands and see that the right operand is bigger, so [316] it turns the subtraction around and puts a negation out front. Now [317] it can do the subtraction because these operands are just approximations and so they do have a right end. And so [318] the answer is approximate. But it's a lot less work. If we were using [319] a point and quote, three divided by four is point seven five, and minus five divided by six is three quote two point five, and it's an addition so we add, not subtract. And we get three quote point two five. That's the least work, and the answer is exact, but our education has not taught us quote notation, so the answer is unfamiliar, and very few people would be satisfied with it.

[320] Comparing a point and rep with a point a quote, [321] first of all, the point and rep doesn't give you any negatives. A point and quote gives you all rationals. [322] Once you have a rep, you can't do any more arithmetic because you don't have a right end to start the operation. With a point and quote you can do all further arithmetic. [323] With a point a rep, there are duplicate representations. With a point and quote, representations are unique after you get rid of repetitions within repetitions, and excess zeros. This is important for comparisons. [324] Point and rep numbers are easily understood, but point and quote numbers not so much. [325] Point and rep is easily approximated, point and quote makes approximation impossible. So I would say they each have their good and bad aspects.

[326] Now I want to look at the space requirements, which means how many digits are required: for unevaluated expressions, expressions evaluated with a rep, and expressions evaluated with a quote. [327] I'll start with this example. One over seven is nice and short, but when you evaluate it, it's longer. That's in decimal, but [328] in binary, they're all about the same length. [329] Here's a worse example. In binary [330] it's not quite so bad, but still, unevaluated is the winner. [331] Here's an example the other way round. Since it's a negative number, there's no rep version, but the quote version is one third as long. In [332] binary, these are not the same numbers, but it's the same pattern, and the quote expression is one third as long as the unevaluated expression. [333] This same pattern always gives this result. That's decimal, but [334] binary also gives the same comparison. Sometimes unevaluated wins, sometimes evaluated wins. So which is best? To try to answer, [335] I looked at the one hundred and eighty thousand shortest unevaluated numbers, in binary, but

just positive numbers so that rep can play too, and I found that rep and quote take the same space as each other, but they take more than twice as much space as unevaluated. But this isn't fair because I'm looking at the shortest unevaluated numbers. So I [336] looked at the one and a half million shortest quote numbers, and I found that the unevaluated form, even with the dividend and divisor reduced to lowest terms, took about twice as much space. So which is better? I really don't know.

[337] In conclusion, finally, I say that each form of expression has its merits, so it's good to know all of them.

I hope you found this video interesting, and I hope you learned something from it.