111 Without using the Bounding Laws, prove

(a) $\forall i\cdot L\,i \le m \;=\; \Uparrow L \le m$

(b) $\exists i\cdot L\,i \le m \;=\; \Downarrow L \le m$

After trying the question, scroll down to the solution.

(a) $\forall i\cdot L\,i \le m \;=\; \Uparrow L \le m$

§ If $\#L{=}0$ then $\forall$ has an empty domain and $\Uparrow L = -\infty$, so the theorem is obvious. Now assume $\#L{>}0$. We prove the equation by proving two implications.

$$
\begin{array}{lll}
 & \forall i\cdot L\,i \le m & \text{Extreme Law} \\
= & (\forall i\cdot L\,i \le m) \wedge (\exists j\cdot L\,j = \Uparrow L) & \text{distribution} \\
= & \exists j\cdot\ (\forall i\cdot L\,i \le m) \wedge\ L\,j = \Uparrow L & \text{specialization and monotonicity} \\
\Longrightarrow & \exists j\cdot\ L\,j \le m\ \wedge\ L\,j = \Uparrow L & \\
= & \exists j\cdot\ \Uparrow L \le m\ \wedge\ L\,j = \Uparrow L & \\
\Longrightarrow & \Uparrow L \le m &
\end{array}
$$

Now the other way: $\forall i\cdot L\,i \le m \;\Longleftarrow\; \Uparrow L \le m$.

$$
\begin{array}{lll}
 & \forall m\cdot \Uparrow L \le m\ \Rightarrow\ \forall i\cdot L\,i \le m & \text{distribution} \\
= & \forall i\cdot \forall m\cdot \Uparrow L \le m\ \Rightarrow\ L\,i \le m & \text{Connection Law (Galois)} \\
= & \forall i\cdot L\,i \le \Uparrow L & \text{Extreme Law} \\
= & \top &
\end{array}
$$

(b) $\exists i\cdot L\,i \le m \;=\; \Downarrow L \le m$

§ The proof is just like part (a).