218 (local minimum) You are given a list L of at least 3 numbers such that $L\,0 \ge L\,1$ and $L\,(\#L-2) \le L\,(\#L-1)$. A local minimum is an interior index i: $1,..\#L-1$ such that

$$L\,(i-1) \ge L\,i \le L\,(i+1)$$

Write a program to find a local minimum of L .

After trying the question, scroll down to the solution.

§ Specification P is defined as

$$P \;=\; i'\!: 1,..\#L{-}1 \;\land\; L(i'{-}1) \ge L\,i' \le L(i'{+}1)$$

Here is a linear search solution. Let i be a natural variable.

$$P \;\Leftarrow\; i{:=}\,1.\; Q$$

$$Q \;\Leftarrow\; \textbf{if } L\,i \le L(i{+}1) \textbf{ then } ok \textbf{ else } i{:=}\,i{+}1.\; Q \textbf{ fi}$$

Now we need to define specification Q. Here is the first attempt: make it just like P except change the 1 to i.

$$Q \;=\; i'\!: i,..\#L{-}1 \;\land\; L(i'{-}1) \ge L\,i' \le L(i'{+}1)$$

Proof of P refinement:

$$\begin{array}{lll}
 & i{:=}\,1.\; Q & \text{expand } Q \\
= & i{:=}\,1.\; i'\!: i,..\#L{-}1 \;\land\; L(i'{-}1) \ge L\,i' \le L(i'{+}1) & \text{substitution law} \\
= & i'\!: 1,..\#L{-}1 \;\land\; L(i'{-}1) \ge L\,i' \le L(i'{+}1) & \\
= & P &
\end{array}$$

Proof of Q refinement, first case:

$$\begin{array}{lll}
 & L\,i \le L(i{+}1) \;\land\; ok & \text{expand } ok \\
= & L\,i \le L(i{+}1) \;\land\; i'{=}i & \text{portation}
\end{array}$$

This is not quite enough to imply Q. We also need $i < \#L{-}1$ and $L(i{-}1) \ge L\,i$. So weaken Q.

$$Q \;=\; i < \#L{-}1 \;\land\; L(i{-}1) \ge L\,i \;\Rightarrow\; i'\!: i,..\#L{-}1 \;\land\; L(i'{-}1) \ge L\,i' \le L(i'{+}1)$$

Now I have to redo the previous proof.

Proof of P refinement:

$$\begin{array}{lll}
 & i{:=}\,1.\; Q & \text{expand } Q \\
= & i{:=}\,1.\; i < \#L{-}1 \;\land\; L(i{-}1) \ge L\,i \;\Rightarrow\; i'\!: i,..\#L{-}1 \;\land\; L(i'{-}1) \ge L\,i' \le L(i'{+}1) & \\
 & & \text{substitution law} \\
= & 1 < \#L{-}1 \;\land\; L\,0 \ge L\,1 \;\Rightarrow\; i'\!: 1,..\#L{-}1 \;\land\; L(i'{-}1) \ge L\,i' \le L(i'{+}1) & \\
 & & \text{given } \#L \ge 3 \text{ and given } L\,0 \ge L\,1 \\
= & \top \;\Rightarrow\; i'\!: 1,..\#L{-}1 \;\land\; L(i'{-}1) \ge L\,i' \le L(i'{+}1) & \text{identity} \\
= & i'\!: 1,..\#L{-}1 \;\land\; L(i'{-}1) \ge L\,i' \le L(i'{+}1) & \\
= & P &
\end{array}$$

Proof of Q refinement, first case:

$$\begin{array}{lll}
 & Q \;\Leftarrow\; L\,i \le L(i{+}1) \;\land\; ok & \text{expand } Q \text{ and } ok \\
= & \quad (i < \#L{-}1 \;\land\; L(i{-}1) \ge L\,i \;\Rightarrow\; i'\!: i,..\#L{-}1 \;\land\; L(i'{-}1) \ge L\,i' \le L(i'{+}1)) & \\
 & \Leftarrow\; L\,i \le L(i{+}1) \;\land\; i'{=}i & \text{portation} \\
= & \quad L\,i \le L(i{+}1) \;\land\; i'{=}i \;\land\; i < \#L{-}1 \;\land\; L(i{-}1) \ge L\,i & \\
 & \Rightarrow\; i'\!: i,..\#L{-}1 \;\land\; L(i'{-}1) \ge L\,i' \le L(i'{+}1) & \text{context} \\
= & \top &
\end{array}$$

Proof of Q refinement, last case:

$$\begin{array}{lll}
 & Q \;\Leftarrow\; L\,i > L(i{+}1) \;\land\; (i{:=}\,i{+}1.\; Q) & \text{expand first } Q \text{ and portation} \\
= & \quad i < \#L{-}1 \;\land\; L(i{-}1) \ge L\,i \;\land\; L\,i > L(i{+}1) \;\land\; (i{:=}\,i{+}1.\; Q) & \\
 & \Rightarrow\; i'\!: i,..\#L{-}1 \;\land\; L(i'{-}1) \ge L\,i' \le L(i'{+}1) & \text{expand remaining } Q
\end{array}$$

$$= \qquad i<\#L-1 \ \land\ L(i-1)\ge L\,i \ \land\ L\,i > L(i+1)$$
$$\land\ (i{:=}\,i+1.\ \ i<\#L-1 \ \land\ L(i-1)\ge L\,i \ \Rightarrow\ i'\!: i,..\#L-1 \ \land\ L(i'-1)\ge L\,i' \le L(i'+1))$$
$$\Rightarrow\ i'\!: i,..\#L-1 \ \land\ L(i'-1)\ge L\,i' \le L(i'+1)$$
substitution law
$$= \qquad i<\#L-1 \ \land\ L(i-1)\ge L\,i \ \land\ L\,i > L(i+1)$$
$$\land\ (i<\#L-2 \ \land\ L\,i\ge L(i+1) \ \Rightarrow\ i'\!: i+1,..\#L-1 \ \land\ L(i'-1)\ge L\,i' \le L(i'+1))$$
$$\Rightarrow\ i'\!: i,..\#L-1 \ \land\ L(i'-1)\ge L\,i' \le L(i'+1)$$

Here's my informal thinking. I see that the consequent of the inner implication

$$i'\!: i+1,..\#L-1 \ \land\ L(i'-1)\ge L\,i' \le L(i'+1)$$

implies the main consequent

$$i'\!: i,..\#L-1 \ \land\ L(i'-1)\ge L\,i' \le L(i'+1)$$

So I need to get rid of the antecedent of the inner implication. I can discharge it if I can show

$$i<\#L-1 \ \land\ L(i-1)\ge L\,i \ \land\ L\,i > L(i+1) \ \Rightarrow\ i<\#L-2 \ \land\ L\,i\ge L(i+1)$$

which is the same as

$$(i<\#L-2 \ \lor\ i=\#L-2) \ \land\ L(i-1)\ge L\,i \ \land\ L\,i > L(i+1) \ \Rightarrow\ i<\#L-2 \ \land\ L\,i\ge L(i+1)$$

which is the same as

$$(i<\#L-2 \ \land\ L(i-1)\ge L\,i \ \land\ L\,i > L(i+1) \ \Rightarrow\ i<\#L-2 \ \land\ L\,i\ge L(i+1))$$
$$\land\ (i=\#L-2 \ \land\ L(i-1)\ge L\,i \ \land\ L\,i > L(i+1) \ \Rightarrow\ i<\#L-2 \ \land\ L\,i\ge L(i+1))$$

The top line is $\top$. So let's work on the bottom line.

$$i=\#L-2 \ \land\ L(i-1)\ge L\,i \ \land\ L\,i > L(i+1) \ \Rightarrow\ i<\#L-2 \ \land\ L\,i\ge L(i+1)$$
context
$$= \quad i=\#L-2 \ \land\ L(i-1)\ge L\,i \ \land\ L(\#L-2) > L(\#L-1) \ \Rightarrow\ i<\#L-2 \ \land\ L\,i\ge L(i+1)$$

But we are given $L(\#L-2)\le L(\#L-1)$. So the antecedent is $\bot$. So the bottom line is $\top$.
So that's the plan. Now write it formally. Resuming from where I left off,

$$= \qquad \underline{i<\#L-1} \ \land\ L(i-1)\ge L\,i \ \land\ L\,i > L(i+1)$$
rewrite underlined bit
$$\land\ (i<\#L-2 \ \land\ L\,i\ge L(i+1) \ \Rightarrow\ i'\!: i+1,..\#L-1 \ \land\ L(i'-1)\ge L\,i' \le L(i'+1))$$
$$\Rightarrow\ i'\!: i,..\#L-1 \ \land\ L(i'-1)\ge L\,i' \le L(i'+1)$$
$$= \qquad (i<\#L-2 \ \lor\ i=\#L-2) \ \land\ L(i-1)\ge L\,i \ \land\ L\,i > L(i+1)$$
distribution this line
$$\land\ (i<\#L-2 \ \land\ L\,i\ge L(i+1) \ \Rightarrow\ i'\!: i+1,..\#L-1 \ \land\ L(i'-1)\ge L\,i' \le L(i'+1))$$
$$\Rightarrow\ i'\!: i,..\#L-1 \ \land\ L(i'-1)\ge L\,i' \le L(i'+1)$$
$$= \qquad (\quad i<\#L-2 \ \land\ L(i-1)\ge L\,i \ \land\ L\,i > L(i+1)$$
$$\lor\ \ i=\#L-2 \ \land\ L(i-1)\ge L\,i \ \land\ L\,i > L(i+1)\ )$$
context this line
$$\land\ (i<\#L-2 \ \land\ L\,i\ge L(i+1) \ \Rightarrow\ i'\!: i+1,..\#L-1 \ \land\ L(i'-1)\ge L\,i' \le L(i'+1))$$
$$\Rightarrow\ i'\!: i,..\#L-1 \ \land\ L(i'-1)\ge L\,i' \le L(i'+1)$$
$$= \qquad (\quad i<\#L-2 \ \land\ L(i-1)\ge L\,i \ \land\ L\,i > L(i+1)$$
$$\lor\ \ i=\#L-2 \ \land\ L(i-1)\ge L\,i \ \land\ L(\#L-2) > L(\#L-1)\ )$$
$$\land\ (i<\#L-2 \ \land\ L\,i\ge L(i+1) \ \Rightarrow\ i'\!: i+1,..\#L-1 \ \land\ L(i'-1)\ge L\,i' \le L(i'+1))$$
$$\Rightarrow\ i'\!: i,..\#L-1 \ \land\ L(i'-1)\ge L\,i' \le L(i'+1)$$
We are given $L(\#L-2)\le L(\#L-1)$
$$= \qquad (\quad i<\#L-2 \ \land\ L(i-1)\ge L\,i \ \land\ L\,i > L(i+1)$$
$$\lor\ \ i=\#L-2 \ \land\ L(i-1)\ge L\,i \ \land\ \bot\ )$$
$$\land\ (i<\#L-2 \ \land\ L\,i\ge L(i+1) \ \Rightarrow\ i'\!: i+1,..\#L-1 \ \land\ L(i'-1)\ge L\,i' \le L(i'+1))$$
$$\Rightarrow\ i'\!: i,..\#L-1 \ \land\ L(i'-1)\ge L\,i' \le L(i'+1)$$
base, identity
$$= \qquad i<\#L-2 \ \land\ L(i-1)\ge L\,i \ \land\ L\,i > L(i+1)$$
$$\land\ (i<\#L-2 \ \land\ L\,i\ge L(i+1) \ \Rightarrow\ i'\!: i+1,..\#L-1 \ \land\ L(i'-1)\ge L\,i' \le L(i'+1))$$
$$\Rightarrow\ i'\!: i,..\#L-1 \ \land\ L(i'-1)\ge L\,i' \le L(i'+1)$$
discharge
$$= \qquad i<\#L-2 \ \land\ L(i-1)\ge L\,i \ \land\ L\,i > L(i+1)$$
$$\land\ i'\!: i+1,..\#L-1 \ \land\ L(i'-1)\ge L\,i' \le L(i'+1)$$
$$\Rightarrow\ i'\!: i,..\#L-1 \ \land\ L(i'-1)\ge L\,i' \le L(i'+1)$$
specialization
$$= \qquad \top$$

There may be a better solution than linear search.