260 (machine multiplication) Given two natural numbers, write a program to find their product using only addition, subtraction, doubling, halving, test for even, and test for zero, but not multiplication or division.

After trying the question, scroll down to the solution.

§ For a solution with linear time, use the identity

$$x \times y = (x-1) \times y + y$$

For a solution with logarithmic time, use the identities

$$x \times y = x/2 \times y \times 2 \quad \text{(for even } x \text{)}$$

$$x \times y = (x-1)/2 \times y \times 2 + y \quad \text{(for odd } x \text{)}$$

Let all variables be natural.

$x:= x \times y \ \Leftarrow$ **if** $x=0$ **then** *ok*
else if *even x* **then** $x:= x/2.\ x:= x \times y.\ x:= x \times 2$
else $x:= (x-1)/2.\ x:= x \times y.\ x:= x \times 2.\ x:= x+y$ **fi fi**

Note that in the solution, the occurrences of $x:= x \times y$ are recursive calls. Note also that in the usual binary representation of natural numbers, $x:= x \times 2$ is just shift left, and both $x:= x/2$ (for even x) and $x:= (x-1)/2$ (for odd x) are just shift right. The execution time is **if** $x=0$ **then** 0 **else** 1 + *floor* (*log x*) **fi**.

Here is another solution in which the recursive calls can be implemented as branches. Let *nat* variables a and b have the given numbers as their initial values, and let *nat* variable c have their product as its final value.

$c' = a \times b \ \Leftarrow \ c:= 0.\ c' = c + a \times b$

$c' = c + a \times b \ \Leftarrow$ **if** $a=0$ **then** *ok*
else if *even a* **then** $a:= a/2.\ b:= b \times 2.\ c' = c + a \times b$
else $c:= c+b.\ a:= a-1.\ c' = c + a \times b$ **fi fi**

with execution time **if** $a=0$ **then** 0 **else** 1 + *floor* (*log a*) **fi**

Both of these solutions can be improved by testing for evenness before testing for zeroness. If a is not even, then it's not zero, and we save a test each iteration. Here's the second program with this improvement.

$c' = a \times b \ \Leftarrow \ c:= 0.\ c' = c + a \times b$

$c' = c + a \times b \ \Leftarrow$ **if** *even a*
then if $a=0$ **then** *ok*
else $a:= a/2.\ b:= b \times 2.\ c' = c + a \times b$ **fi**
else $c:= c+b.\ a:= a-1.\ c' = c + a \times b$ **fi**