290 (factorial space) We can compute $x:= n!$ (factorial) as follows.

$$x:= n! \ \Leftarrow\ \textbf{if } n=0 \textbf{ then } x:= 1 \textbf{ else } n:= n-1.\ x:= n!.\ n:= n+1.\ x:= x\times n \textbf{ fi}$$

Each call $x:= n!$ pushes a return address onto a stack, and each return pops an address from the stack. Add a space variable s and a maximum space variable m, with appropriate assignments to them in the program. Find and prove an upper bound on the maximum space used.

After trying the question, scroll down to the solution.

§ $s \le m \le s{+}n \;\Rightarrow\; m' = s{+}n \;\Longleftarrow$

$\quad$ **if** $n{=}0$ **then** $x{:=}\ 1$

$\quad$ **else** $n{:=}\ n{-}1.$

$\qquad s{:=}\ s{+}1.\;\; m{:=}\ m{\uparrow}s.\;\; s \le m \le s{+}n \;\Rightarrow\; m' = s{+}n.\;\; s{:=}\ s{-}1.$

$\qquad n{:=}\ n{+}1.\;\; x{:=}\ x{\times}n$ **fi**

Proof: by cases. First case:

$(n{=}0 \wedge (x{:=}\ 1) \;\Rightarrow\; (s \le m \le s{+}n \;\Rightarrow\; m' = s{+}n))$ — portation and expand

$= \quad n{=}0 \wedge x'{=}1 \wedge n'{=}n \wedge s'{=}s \wedge m'{=}m \wedge s{\le}m{\le}s{+}n \;\Rightarrow\; m' = s{+}n$ — context

$= \quad n{=}0 \wedge x'{=}1 \wedge n'{=}n \wedge s'{=}s \wedge m'{=}m \wedge s{\le}m{\le}s{+}n \;\Rightarrow\; \top$ — base

$= \quad \top$

Second case:

$(\quad s{\le}m{\le}s{+}n \;\Rightarrow\; m' = s{+}n$

$\Longleftarrow\; (n{\ne}0 \wedge (\; n{:=}\ n{-}1.\;\; s{:=}\ s{+}1.$

$\qquad m{:=}\ m{\uparrow}s.\;\; s{\le}m{\le}s{+}n \;\Rightarrow\; m' = s{+}n.\;\; s{:=}\ s{-}1.$

$\qquad n{:=}\ n{+}1.\;\; x{:=}\ x{\times}n))$ — portation

$= \quad n{\ne}0 \wedge s{\le}m{\le}s{+}n$

$\quad \wedge\; (\, n{:=}\ n{-}1.\;\; s{:=}\ s{+}1.\;\; m{:=}\ m{\uparrow}s.\;\; s{\le}m{\le}s{+}n \;\Rightarrow\; m' = s{+}n.$

$\qquad s{:=}\ s{-}1.\;\; n{:=}\ n{+}1.\;\; x{:=}\ x{\times}n)$

$\Rightarrow\; m' = s{+}n$

three substitutions; expand final assignment and two more substitutions

$= \quad n{\ne}0 \wedge s{\le}m{\le}s{+}n$

$\quad \wedge\; (\, s{+}1 \le m{\uparrow}(s{+}1) \le s{+}1{+}n{-}1 \;\Rightarrow\; m' = s{+}1{+}n{-}1.$

$\qquad x' = x{\times}(n{+}1) \wedge n'{=}n{+}1 \wedge s'{=}s{-}1 \wedge m'{=}m)$

$\Rightarrow\; m' = s{+}n$ — simplify $+1{-}1$ twice

$= \quad n{\ne}0 \wedge s{\le}m{\le}s{+}n$

$\quad \wedge\; (\, s{+}1 \le m{\uparrow}(s{+}1) \le s{+}n \;\Rightarrow\; m' = s{+}n.$

$\qquad x' = x{\times}(n{+}1) \wedge n'{=}n{+}1 \wedge s'{=}s{-}1 \wedge m'{=}m)$

$\Rightarrow\; m' = s{+}n$ — eliminate sequential composition

$= \quad n{\ne}0 \wedge s{\le}m{\le}s{+}n$

$\quad \wedge\; (\, \exists n'', m'', s'', x''\cdot\;\; (s{+}1 \le m{\uparrow}(s{+}1) \le s{+}n \;\Rightarrow\; m'' = s{+}n)$

$\qquad\qquad \wedge\;\; x' = x{\times}(n{+}1) \wedge n'{=}n{+}1 \wedge s'{=}s{-}1 \wedge m'{=}m)$

$\Rightarrow\; m' = s{+}n$ — some arithmetic

$= \quad n{\ne}0 \wedge s{\le}m{\le}s{+}n$

$\quad \wedge\; (\, \exists n'', m'', s'', x''\cdot\;\; (s{+}1 \le m{\uparrow}(s{+}1) \le s{+}n \;\Rightarrow\; m'' = s{+}n)$

$\qquad\qquad \wedge\;\; x'' = x'/(n''{+}1) \wedge n''{=}n'{-}1 \wedge s''{=}s'{+}1 \wedge m''{=}m')$

$\Rightarrow\; m' = s{+}n$ — one-point 4 times

$= \quad n{\ne}0 \wedge s{\le}m{\le}s{+}n$

$\quad \wedge\; (\, s{+}1 \le m{\uparrow}(s{+}1) \le s{+}n \;\Rightarrow\; m' = s{+}n)$

$\Rightarrow\; m' = s{+}n$ — $s{+}1 \le m{\uparrow}(s{+}1)$ simplifies to $\top$

and in the context $n{\ne}0 \wedge s{\le}m{\le}s{+}n$ we simplify $m{\uparrow}(s{+}1) \le s{+}n$ to $\top$

$= \quad n{\ne}0 \wedge s{\le}m{\le}s{+}n$

$\quad \wedge\; (\, \top \;\Rightarrow\; m' = s{+}n)$

$\Rightarrow\; m' = s{+}n$ — identity and specialize

$= \quad \top$