300 Let all variables be integer except L is a list of integers. What is the exact precondition

(a) for $x'+y' > 8$ to be refined by $x:= 1$

(b) for $x'=1$ to be refined by $x:= 1$

(c) for $x'=2$ to be refined by $x:= 1$

(d) for $x'=y$ to be refined by $y:= 1$

(e) for $x' \ge y'$ to be refined by $x:= y+z$

(f) for $y'+z' \ge 0$ to be refined by $x:= y+z$

(g) for $x' \le 1 \lor x' \ge 5$ to be refined by $x:= x+1$

(h) for $x'<y' \land \exists x\cdot L\,x < y'$ to be refined by $x:= 1$

(i) for $\exists y\cdot L\,y < x'$ to be refined by $x:= y+1$

(j) for $L'\,3 = 4$ to be refined by $L:= i\rightarrow 4 \mid L$

(k) for $x'=a$ to be refined by **if** $a > b$ **then** $x:= a$ **else** ok **fi**

(l) for $x'=y \land y'=x$ to be refined by $(z:= x.\ x:= y.\ y:= z)$

(m) for $a\times x'^{\,2} + b\times x' + c = 0$ to be refined by $(x:= a\times x + b.\ x:= -x/a)$

(n) for $f' = n'!$ to be refined by $(n:= n+1.\,f:= f\times n)$ where n is natural and $!$ is factorial.

(o) for $7 \le c' < 28 \land odd\ c'$ to be refined by $(a:= b-1.\ b:= a+3.\ c:= a+b)$

(p) for $s' = \Sigma L\,[0;..i']$ to be refined by $(s:= s + L\,i.\ i:= i+1)$

(q) for $x'>5$ to be refined by $x'\colon x+(1,2)$

(r) for $x'>0$ to be refined by $x'\colon x+(-1, 1)$

After trying the question, scroll down to the solution.

(a) for $x'+y' > 8$ to be refined by $x:= 1$

§ $\forall x', y' \cdot x'+y' > 8 \Leftarrow (x:= 1)$

$=$ $\forall x', y' \cdot x'+y' > 8 \Leftarrow x'=1 \wedge y'=y$ one-point twice

$=$ $1+y > 8$

$=$ $y>7$

(b) for $x'=1$ to be refined by $x:= 1$

§ $\forall x', y' \cdot x'=1 \Leftarrow (x:= 1)$

$=$ $\forall x', y' \cdot x'=1 \Leftarrow x'=1$ one-point

$=$ $1=1$

$=$ $\top$

(c) for $x'=2$ to be refined by $x:= 1$

§ $\forall x', y' \cdot x'=2 \Leftarrow (x:= 1)$

$=$ $\forall x', y' \cdot x'=2 \Leftarrow x'=1$ one-point

$=$ $1=2$

$=$ $\perp$

(d) for $x'=y$ to be refined by $y:= 1$

§ $\forall x', y' \cdot x'=y \Leftarrow (y:= 1)$

$=$ $\forall x', y' \cdot x'=y \Leftarrow x'=x \wedge y'=1$ one-point twice

$=$ $x=y$

(e) for $x'\ge y'$ to be refined by $x:= y+z$

§ $\forall x', y', z' \cdot x'\ge y' \Leftarrow (x:= y+z)$ expand assignment

$=$ $\forall x', y', z' \cdot x'\ge y' \Leftarrow x' = y+z \wedge y'=y \wedge z'=z$ one-point, three times

$=$ $y+z \ge y$ arithmetic

$=$ $z\ge 0$

(f) for $y'+z' \ge 0$ to be refined by $x:= y+z$

§ $\forall x', y', z' \cdot y'+z' \ge 0 \Leftarrow (x:= y+z)$

$=$ $\forall x', y', z' \cdot y'+z' \ge 0 \Leftarrow x'=y+z \wedge y'=y \wedge z'=z$ One-point, 3 times

$=$ $y+z \ge 0$

(g) for $x'\le 1 \vee x'\ge 5$ to be refined by $x:= x+1$

§ $\forall x' \cdot (x'\le 1 \vee x'\ge 5 \Leftarrow x:= x+1)$

$=$ $\forall x' \cdot (x'\le 1 \vee x'\ge 5 \Leftarrow x' = x+1)$ one-point

$=$ $x+1\le 1 \vee x+1\ge 5$

$=$ $x\le 0 \vee x\ge 4$

$=$ $\neg\, x\colon 1,..4$

(h) for $x'<y' \wedge \exists x\cdot L\,x < y'$ to be refined by $x:= 1$

§ $\forall x', y', L' \cdot\ x'<y' \wedge (\exists x\cdot L\,x < y') \Leftarrow (x:= 1)$

$=$ $\forall x', y', L' \cdot\ x'<y' \wedge (\exists x\cdot L\,x < y') \Leftarrow x'=1 \wedge y'=y \wedge L'=L$ one-point three times

$=$ $1<y \wedge \exists x\cdot L\,x < y$

(i) for $\exists y\cdot L\,y < x'$ to be refined by $x:= y+1$

§ $\forall x', y', L' \cdot (\exists y\cdot L\,y < x') \Leftarrow (x:= y+1)$ expand assignment

$=$ $\forall x', y', L' \cdot (\exists y\cdot L\,y < x') \Leftarrow x'=y+1 \wedge y'=y \wedge L'=L$ rename

$=$ $\forall x', y', L' \cdot (\exists z\cdot L\,z < x') \Leftarrow x'=y+1 \wedge y'=y \wedge L'=L$ one-point three times

$=$ $\exists z\cdot L\,z < y+1$

(j) for $L'\,3 = 4$ to be refined by $L{:=}\, i{\to}4 \mid L$

§ $\forall L', i'\cdot L'\,3 = 4 \Leftarrow (L{:=}\, i{\to}4 \mid L)$

$=$ $\forall L', i'\cdot L'\,3 = 4 \Leftarrow L' = i{\to}4 \mid L \;\land\; i'{=}i$ one-point twice

$=$ $(i{\to}4 \mid L)\,3 = 4$

$=$ $i{=}3 \;\lor\; L\,3 = 4$

(k) for $x'{=}a$ to be refined by **if** $a > b$ **then** $x{:=}\, a$ **else** *ok* **fi**

§ $\forall x', a', b'\cdot x'{=}a \Leftarrow$ **if** $a > b$ **then** $x{:=}\, a$ **else** *ok* **fi** replace **if** , $:=$, and *ok*

$=$ $\forall x', a', b'\cdot x'{=}a \Leftarrow (a{>}b \land x'{=}a \land a'{=}a \land b'{=}b) \lor (a{\le}b \land x'{=}x \land a'{=}a \land b'{=}b)$ antidist

$=$ $\forall x', a', b'\cdot (x'{=}a \Leftarrow a{>}b \land x'{=}a \land a'{=}a \land b'{=}b) \land (x'{=}a \Leftarrow a{\le}b \land x'{=}x \land a'{=}a \land b'{=}b)$

splitting

$=$ $(\forall x', a', b'\cdot x'{=}a \Leftarrow a{>}b \land x'{=}a \land a'{=}a \land b'{=}b)$

$\land\; (\forall x', a', b'\cdot x'{=}a \Leftarrow a{\le}b \land x'{=}x \land a'{=}a \land b'{=}b)$

specialization and identity; one-point

$=$ $x{=}a \Leftarrow a{\le}b$

(l) for $x'{=}y \land y'{=}x$ to be refined by $(z{:=}\, x.\;\; x{:=}\, y.\;\; y{:=}\, z)$

§ $\forall x', y', z'\cdot x'{=}y \land y'{=}x \;\Leftarrow\; (z{:=}\, x.\;\; x{:=}\, y.\;\; y{:=}\, z)$

$=$ $\forall x', y', z'\cdot x'{=}y \land y'{=}x \;\Leftarrow\; (z{:=}\, x.\;\; x{:=}\, y.\;\; x'{=}x \land y'{=}z \land z'{=}z)$ Substitution Law

$=$ $\forall x', y', z'\cdot x'{=}y \land y'{=}x \;\Leftarrow\; (z{:=}\, x.\;\; x'{=}y \land y'{=}z \land z'{=}z)$ Substitution Law

$=$ $\forall x', y', z'\cdot x'{=}y \land y'{=}x \;\Leftarrow\; x'{=}y \land y'{=}x \land z'{=}x$ One-point, 3 times

$=$ $y{=}y \;\land\; x{=}x$

$=$ $\top$

(m) for $a{\times}x'^{\,2} + b{\times}x' + c = 0$ to be refined by $(x{:=}\, a{\times}x + b.\;\; x{:=}\, -x/a)$

§ $\forall x'\cdot a{\times}x'^{\,2} + b{\times}x' + c = 0 \;\Leftarrow\; (x{:=}\, a{\times}x + b.\;\; x{:=}\, -x/a)$ replace final assignment

$=$ $\forall x'\cdot a{\times}x'^{\,2} + b{\times}x' + c = 0 \;\Leftarrow\; (x{:=}\, a{\times}x + b.\;\; x' = -x/a)$ substitution law

$=$ $\forall x'\cdot a{\times}x'^{\,2} + b{\times}x' + c = 0 \;\Leftarrow\; x' = -(a{\times}x + b)/a$ one point

$=$ $a{\times}(-(a{\times}x + b)/a)^2 + b{\times}(-(a{\times}x + b)/a) + c = 0$ This is the exact precondition.

But we can simplify it if we allow a sufficient precondition answer:

$\Leftarrow$ $a{\ne}0 \;\land\; a{\times}x^2 + b{\times}x + c = 0$

(n) for $f' = n'!$ to be refined by $(n{:=}\, n{+}1.\;\; f{:=}\, f{\times}n)$ where n is natural and $!$ is factorial.

§ $\forall f', n'\cdot f' = n'! \;\Leftarrow\; (n{:=}\, n{+}1.\;\; f{:=}\, f{\times}n)$ expand last assignment

$=$ $\forall f', n'\cdot f' = n'! \;\Leftarrow\; (n{:=}\, n{+}1.\;\; f'{=}f{\times}n \;\land\; n'{=}n)$ substitution law

$=$ $\forall f', n'\cdot f' = n'! \;\Leftarrow\; f'{=}f{\times}(n{+}1) \;\land\; n'{=}n{+}1$ one-point twice

$=$ $f{\times}(n{+}1) = (n{+}1)!$ definition of $!$

$=$ $f{\times}(n{+}1) = n!{\times}(n{+}1)$ cancellation

$=$ $f = n!$

(o) for $7{\le}c'{<}28 \;\land\; odd\; c'$ to be refined by $(a{:=}\, b{-}1.\;\; b{:=}\, a{+}3.\;\; c{:=}\, a{+}b)$

§ $\forall a', b', c'\cdot 7{\le}c'{<}28 \;\land\; odd\; c' \;\Leftarrow\; (a{:=}\, b{-}1.\;\; b{:=}\, a{+}3.\;\; c{:=}\, a{+}b)$ expand last asmt

$=$ $\forall a', b', c'\cdot 7{\le}c'{<}28 \;\land\; odd\; c' \;\Leftarrow\; (a{:=}\, b{-}1.\;\; b{:=}\, a{+}3.\;\; a'{=}a \land b'{=}b \land c'{=}a{+}b)$

substitution law twice

$=$ $\forall a', b', c'\cdot 7{\le}c'{<}28 \;\land\; odd\; c' \;\Leftarrow\; a'{=}b{-}1 \land b'{=}b{+}2 \land c'{=}2{\times}b{+}1$ one-pt 3 times

$=$ $7 \le 2{\times}b{+}1 < 28 \;\land\; odd\,(2{\times}b{+}1)$

$=$ $3 \le b < 14$

(p) for $s' \;=\; \Sigma\, L\,[0;..i']$ to be refined by $(s{:=}\, s + L\,i.\;\; i{:=}\, i{+}1)$

§ $\forall s', i', L'\cdot (s' = \Sigma\, L\,[0;..i']) \Leftarrow s'{=}s + L\,i \land i'{=}i{+}1 \land L'{=}L$

$=$ $s + L\,i = \Sigma\, L\,[0;..i{+}1]$

$=$ $s \;=\Sigma\, L\,[0;..i]$

(q) for $x'>5$ to be refined by $x': x+(1,2)$

§ $\forall x'\cdot x'>5 \;\Leftarrow\; x': x+(1,2)$ — $+$ distributes over $,$

$=$ $\forall x'\cdot x'>5 \;\Leftarrow\; x': x+1, x+2$ — compound axiom

$=$ $\forall x'\cdot x'>5 \;\Leftarrow\; x': x+1 \vee x': x+2$ — elementary axiom twice

$=$ $\forall x'\cdot x'>5 \;\Leftarrow\; x' = x+1 \;\vee\; x' = x+2$ — antidistribution

$=$ $\forall x'\cdot (x'>5 \;\Leftarrow\; x' = x+1) \wedge (x'>5 \;\Leftarrow\; x' = x+2)$ — distribution

$=$ $(\forall x'\cdot x'>5 \;\Leftarrow\; x' = x+1) \wedge (\forall x'\cdot x'>5 \;\Leftarrow\; x'= x+2)$ — one-point twice

$=$ $x+1 > 5 \;\wedge\; x+2 > 5$ — arithmetic, inclusion because $x>4 \Rightarrow x>3$

$=$ $x>4$

(r) for $x'>0$ to be refined by $x': x+(-1, 1)$

§ $\forall x'\cdot x'>0 \;\Leftarrow\; x': x+(-1, 1)$ — $+$ distributes over $,$

$=$ $\forall x'\cdot x'>0 \;\Leftarrow\; x': x-1, x+1$ — compound axiom

$=$ $\forall x'\cdot x'>0 \;\Leftarrow\; x': x-1 \vee x': x+1$ — elementary axiom twice

$=$ $\forall x'\cdot x'>0 \;\Leftarrow\; x' = x-1 \;\vee\; x' = x+1$ — antidistribution

$=$ $\forall x'\cdot (x'>0 \;\Leftarrow\; x' = x-1) \wedge (x'>0 \;\Leftarrow\; x' = x+1)$ — distribution

$=$ $(\forall x'\cdot x'>0 \;\Leftarrow\; x' = x-1) \wedge (\forall x'\cdot x'>0 \;\Leftarrow\; x'= x+1)$ — one-point twice

$=$ $x-1 > 0 \;\wedge\; x+1 > 0$ — arithmetic

$=$ $x>1 \;\wedge\; x>-1$ — inclusion because $x>1 \Rightarrow x>-1$

$=$ $x>1$