301 For what exact precondition and exact postcondition does the following assignment move integer variable x farther from zero?

(a) $x := x+1$

(b) $x := abs\ (x+1)$

(c)√ $x := x^2$

After trying the question, scroll down to the solution.

(a) $\quad x:= x+1$

$$
\begin{array}{cll}
\S & (\text{the exact precondition for } abs\,x' > abs\,x \text{ to be refined by } x:= x+1\ ) & \\
= & \forall x'\cdot\ abs\,x' > abs\,x\ \Leftarrow\ (x:= x+1) & \\
= & \forall x'\cdot\ abs\,x' > abs\,x\ \Leftarrow\ x' = x+1 & \text{one-point} \\
= & abs\,(x+1) > abs\,x & \\
= & x\ge 0 &
\end{array}
$$

$$
\begin{array}{cll}
 & (\text{the exact postcondition for } abs\,x' > abs\,x \text{ to be refined by } x:= x+1\ ) & \\
= & \forall x\cdot\ abs\,x' > abs\,x\ \Leftarrow\ (x:= x+1) & \\
= & \forall x\cdot\ abs\,x' > abs\,x\ \Leftarrow\ x' = x+1 & \text{one-point} \\
= & abs\,x' > abs\,(x'-1) & \\
= & x'\ge 1 &
\end{array}
$$

(b) $\quad x:= abs\,(x+1)$

$$
\begin{array}{cll}
\S & (\text{the exact precondition for } abs\,x' > abs\,x \text{ to be refined by } x:= abs\,(x+1)\ ) & \\
= & \forall x'\cdot\ abs\,x' > abs\,x\ \Leftarrow\ (x:= abs\,(x+1)) & \\
= & \forall x'\cdot\ abs\,x' > abs\,x\ \Leftarrow\ x' = abs\,(x+1) & \text{one-point} \\
= & abs\,(abs\,(x+1)) > abs\,x & \\
= & abs\,(x+1) > abs\,x & \\
= & x\ge 0 &
\end{array}
$$

$$
\begin{array}{cll}
 & (\text{the exact postcondition for } abs\,x' > abs\,x \text{ to be refined by } x:= abs\,(x+1)\ ) & \\
= & \forall x\cdot\ abs\,x' > abs\,x\ \Leftarrow\ (x:= abs\,(x+1)) & \\
= & \forall x\cdot\ abs\,x' > abs\,x\ \Leftarrow\ x' = abs\,(x+1) & \text{divide domain} \\
= & \quad (\forall x\colon nat\cdot\ abs\,x' > abs\,x\ \Leftarrow\ x' = abs\,(x+1)) & \\
 & \wedge\ \ (\forall x\colon -nat-1\cdot\ abs\,x' > abs\,x\ \Leftarrow\ x' = abs\,(x+1)) & \text{change variable} \\
= & \quad (\forall x\colon nat\cdot\ abs\,x' > abs\,x\ \Leftarrow\ x' = abs\,(x+1)) & \text{remove } abs \text{ twice} \\
 & \wedge\ \ (\forall z\colon nat\cdot\ abs\,x' > abs\,(-z-1)\ \Leftarrow\ x' = abs\,(-z-1+1)) & \text{simplify and remove } abs \\
= & \quad (\forall x\colon nat\cdot\ abs\,x' > x\ \Leftarrow\ x' = x+1) & \text{context} \\
 & \wedge\ \ (\forall z\colon nat\cdot\ abs\,x' > z+1\ \Leftarrow\ x' = z) & \text{context} \\
= & \quad (\forall x\colon nat\cdot\ abs\,(x+1) > x\ \Leftarrow\ x' = x+1) & \text{remove } abs \text{ and simplify} \\
 & \wedge\ \ (\forall z\colon nat\cdot\ abs\,z > z+1\ \Leftarrow\ x' = z) & \text{remove } abs \text{ and simplify} \\
= & \quad (\forall x\colon nat\cdot\ \top\ \Leftarrow\ x' = x+1) & \\
 & \wedge\ \ (\forall z\colon nat\cdot\ \bot\ \Leftarrow\ x' = z) & \\
= & \top\ \ \wedge\ \ (\forall z\colon nat\cdot\ x' \ne z) & \\
= & x'<0 &
\end{array}
$$

(c)√ $\quad x:= x^2$

$$
\begin{array}{cll}
\S & (\text{the exact precondition for } abs\,x' > abs\,x \text{ to be refined by } x:= x^2\ ) & \\
= & \forall x'\cdot\ abs\,x' > abs\,x\ \Leftarrow\ x' = x^2 & \text{One-Point Law} \\
= & abs\,(x^2) > abs\,x & \text{by the arithmetic properties of } abs\,x \text{ and } x^2 \\
= & x\ne -1\ \wedge\ x\ne 0\ \wedge\ x\ne 1 &
\end{array}
$$

$$
\begin{array}{cll}
 & (\text{the exact postcondition for } abs\,x' > abs\,x \text{ to be refined by } x:= x^2\ ) & \\
= & \forall x\colon int\cdot\ abs\,x' > abs\,x\ \Leftarrow\ x' = x^2 & \text{arithmetic: } x^2=(-x)^2 \\
= & \forall x\colon int\cdot\ abs\,x' > abs\,x\ \Leftarrow\ x' = (abs\,x)^2 & \text{change variable} \\
= & \forall y\colon abs\,int\cdot\ abs\,x' > y\ \Leftarrow\ x' = y^2 & \text{context} \\
= & \forall y\colon nat\cdot\ abs\,(y^2) > y\ \Leftarrow\ x' = y^2 & \text{arithmetic: } y^2\ge 0 \\
= & \forall y\colon nat\cdot\ y^2 > y\ \Leftarrow\ x' = y^2 & \text{domain split}
\end{array}
$$

$$
\begin{array}{lll}
= & (\forall y\colon 0\cdot\ y^2 > y \ \Leftarrow\ x' = y^2) \ \wedge\ (\forall y\colon 1\cdot\ y^2 > y \ \Leftarrow\ x' = y^2) & \text{one-point} \\
 & \wedge\ (\forall y\colon nat{+}2\cdot\ y^2 > y \ \Leftarrow\ x' = y^2) & \text{and arithmetic} \\
= & (\bot \Leftarrow x'{=}0) \ \wedge\ (\bot \Leftarrow x'{=}1) \ \wedge\ (\forall y\colon nat{+}2\cdot\ \top \Leftarrow\ x' = y^2) & \\
= & x' {\ne} 0 \wedge x' {\ne} 1 &
\end{array}
$$