330 (majority vote) The problem is to find, in a given list, the majority item (the item that occurs in more than half the places) if there is one. Letting L be the list and m be a variable whose final value is the majority item, prove that the following program solves the problem.

(a)

```
new e: nat := 0·
for i:= 0;..#L
do  if m = L i then e:= e+1
    else  if i = 2×e then m:= L i.  e:= e+1
          else ok fi fi od
```

(b)

```
new s: nat := 0·
for i:= 0;..#L
do  if m = L i then ok
    else  if i = 2×s then m:= L i
          else s:= s+1 fi fi od
```

After trying the question, scroll down to the solution.

(a) $\quad$ **new** $e: nat := 0\cdot$

for $i := 0;..\#L$

do if $m = L\, i$ **then** $e := e+1$

else if $i = 2\times e$ **then** $m := L\, i.\ \ e := e+1$

else ok **fi od**

§ This is a hard question. Let $N\, x\, i \ = \ ¢§j: 0,..i\cdot\ L\, j = x$. So $N\, x\, i$ is the number of occurrences of item x in list L before index i . The problem can be stated formally as

$$\forall x\cdot\ N\, x\, (\#L) > \#L/2 \Rightarrow m'=x$$

or

$$\forall x\cdot\ x{\ne}m' \Rightarrow N\, x\, (\#L) \le \#L/2$$

The invariant we need for the **for**-loop is defined as

$$A\, i \ = \ i{\le}2{\times}e \wedge N\, m\, i \le e \wedge \forall x\cdot\ x{\ne}m \Rightarrow N\, x\, i \le i{-}e$$

We must prove two refinements:

$$\forall x\cdot\ x{\ne}m' \Rightarrow N\, x\, (\#L) \le \#L/2 \ \Leftarrow\ \textbf{var}\ e: nat := 0\cdot\ \ A\, 0 \Rightarrow A'(\#L)$$

$$i: 0,..\#L \ \wedge\ A\, i \ \Rightarrow\ A'(i{+}1) \ \Leftarrow$$

if $m = L\, i$ **then** $e := e+1$

else if $i = 2\times e$ **then** $m := L\, i.\ \ e := e+1$

else ok **fi**

The first refinement is proven as follows:

$\quad$ **new** $e: nat := 0\cdot\ \ A\, 0 \Rightarrow A'(\#L)$

$=\ \ \exists e: 0\cdot\ \exists e': nat\cdot\ \ (0{\le}2{\times}e \wedge N\, m\, 0 \le e \wedge \forall x\cdot\ x{\ne}m \Rightarrow N\, x\, 0 \le 0{-}e)$

$\qquad \Rightarrow\ (\#L{\le}2{\times}e' \wedge N\, m'(\#L) \le e' \wedge \forall x\cdot\ x{\ne}m' \Rightarrow N\, x\, (\#L) \le \#L{-}e')$

$=\ \ \exists e'\cdot\ \ (0{\le}0 \wedge N\, m\, 0 \le 0 \wedge \forall x\cdot\ x{\ne}m \Rightarrow N\, x\, 0 \le 0)$

$\qquad \Rightarrow\ (\#L{\le}2{\times}e' \wedge N\, m'(\#L) \le e' \wedge \forall x\cdot\ x{\ne}m' \Rightarrow N\, x\, (\#L) \le \#L{-}e')$

the antecedent reduces to $\top$.

In the consequent, drop the second conjunct and rewrite the first

$\Rightarrow\ \ \exists e'\cdot\ \ \#L{-}e' \le \#L/2 \ \wedge\ \forall x\cdot\ x{\ne}m' \Rightarrow N\, x\, (\#L) \le \#L{-}e'$

now use the first conjunct to weaken the last

$\Rightarrow\ \ \forall x\cdot\ x{\ne}m' \Rightarrow N\, x\, (\#L) \le \#L/2$

The second refinement to be proven can be broken into three cases. The first case is

$\quad (i: 0,..\#L \ \wedge\ A\, i \ \Rightarrow\ A'(i{+}1) \ \Leftarrow\ m = L\, i \ \wedge\ (e := e+1))$ $\qquad$ portation

$=\ \ m = L\, i \ \wedge\ (e := e+1) \ \wedge\ i: 0,..\#L \ \wedge\ A\, i \ \Rightarrow\ A'(i{+}1)$

$=\ \ \ m = L\, i \ \wedge\ e'{=}e{+}1 \ \wedge\ m'{=}m \ \wedge\ i: 0,..\#L$

$\qquad \wedge\ \ i{\le}2{\times}e \wedge N\, m\, i \le e \wedge (\forall x\cdot\ x{\ne}m \Rightarrow N\, x\, i \le i{-}e)$

$\quad \Rightarrow\ \ i{+}1{\le}2{\times}e' \wedge N\, m'(i{+}1) \le e' \wedge (\forall x\cdot\ x{\ne}m' \Rightarrow N\, x\, (i{+}1) \le i{+}1{-}e')$

in the antecedent we have $m' = m = L\, i$, so $N\, m'(i{+}1) = N\, m\, i + 1$

and for all other x , $N\, x\, i = N\, x\, (i{+}1)$

$=\ \ \top$

The second case is

$\quad (i: 0,..\#L \ \wedge\ A\, i \ \Rightarrow\ A'(i{+}1) \ \Leftarrow\ m{\ne}Li \wedge i{=}2{\times}e \ \wedge\ (m := L\, i.\ \ e := e+1))$ $\qquad$ portation

$=\ \ m \ne L\, i \wedge i{=}2{\times}e \ \wedge\ (m := L\, i.\ \ e := e+1) \ \wedge\ i: 0,..\#L \ \wedge\ A\, i \ \Rightarrow\ A'(i{+}1)$

$=\ \ \ m \ne L\, i \wedge i{=}2{\times}e \ \wedge\ m'{=}L\, i \ \wedge\ e'{=}e{+}1 \ \wedge\ i: 0,..\#L$

$\qquad \wedge\ \ i{\le}2{\times}e \wedge N\, m\, i \le e \wedge (\forall x\cdot\ x{\ne}m \Rightarrow N\, x\, i \le i{-}e)$

$\quad \Rightarrow\ \ i{+}1{\le}2{\times}e' \wedge N\, m'(i{+}1) \le e' \wedge (\forall x\cdot\ x{\ne}m' \Rightarrow N\, x\, (i{+}1) \le i{+}1{-}e')$

simplify antecedent, and use its equations

$=\ \ \ m \ne L\, i = m' \wedge i{=}2{\times}e \ \wedge\ e'{=}e{+}1 \ \wedge\ i: 0,..\#L \ \wedge\ (\forall x\cdot\ N\, x\, i \le e)$

$\quad \Rightarrow\ \ 2{\times}e{+}1 \le 2{\times}(e{+}1) \wedge N(Li)(i{+}1) \le e{+}1 \wedge (\forall x\cdot\ x \ne L\, i \Rightarrow N\, x\, (i{+}1) \le e)$

$=\ \ \top$

The third case is

$\quad (i: 0,..\#L \ \wedge\ A\, i \ \Rightarrow\ A'(i{+}1) \ \Leftarrow\ m \ne L\, i \wedge i{\ne}2{\times}e \ \wedge\ ok)$ $\quad$ portation

$=\ \ m \ne L\, i \wedge i{\ne}2{\times}e \ \wedge\ ok \ \wedge\ i: 0,..\#L \ \wedge\ A\, i \ \Rightarrow\ A'(i{+}1)$

$=\ \ \ m{\ne}Li \wedge i{\ne}2{\times}e \ \wedge\ e'{=}e \ \wedge\ m'{=}m \ \wedge\ i: 0,..\#L$

$\wedge \quad i{\le}2{\times}e \wedge N\ m\ i \le e \wedge (\forall x\cdot\ x{\ne}m \Rightarrow N\ x\ i \le i{-}e)$

$\Rightarrow \quad i{+}1{\le}2{\times}e' \wedge N\ m'(i{+}1) \le e' \wedge (\forall x\cdot\ x{\ne}m' \Rightarrow N\ x\ (i{+}1) \le i{+}1{-}e')$

in the antecedent we have $m' = m \ne L\ i$, so $N\ m'(i{+}1) = N\ m\ i$.

And for all x , $N\ x\ (i{+}1){:}\ N\ x\ i, N\ x\ i + 1$.

$= \quad \top$

Note that the program is correct even though the initial value of variable m is arbitrary.

(b) **new** *s*: *nat* := 0·
for *i*:= 0;..#*L*
do **if** *m* = *L i* **then** *ok*
else if *i* = 2×*s* **then** *m*:= *L i*
else *s*:= *s*+1 **fi od**

§ This is the same as part (a) with $s = i{-}e$.