34 When we defined number expressions, we included complex numbers such as $(-1)^{1/2}$, not because we particularly wanted them, but because it was easier than excluding them. If we were interested in complex numbers, we would find that the number axioms given in Subsection 11.3.2 do not allow us to prove many things we might like to prove. For example, we cannot prove $(-1)^{1/2} \times 0 = 0$. How can the axioms be made strong enough to prove things about complex numbers, but weak enough to leave room for ∞ ?

After trying the question, scroll down to the solution.

§ I suppose there are many ways to do this. Here's one. There are axioms like

$$-\infty<x<\infty \Rightarrow (x+y = x+z \;=\; y=z) \qquad \text{Cancellation}$$

that start off with an antecedent saying "if x is finite then ...", and we have defined the ordering $<$ on the reals but not on the complex numbers. To make the axiom applicable to complex numbers, we need to extend the ordering to complex numbers. But we don't need to determine, for any two complex numbers x and y whether $x<y$. We just need to distinguish finite complex numbers from infinite complex numbers. So let's say

$$-\infty<a<\infty \wedge -\infty<b<\infty \;=\; -\infty < a + b\times i < \infty$$

If both the real part and imaginary part of a complex number are finite, then the complex number is finite, and vice versa. Now these laws apply to complex numbers:

$-\infty<x<\infty \Rightarrow (x+y = x+z \;=\; y=z)$	Cancellation
$-\infty<x \Rightarrow \infty+x = \infty$	Absorption
$x<\infty \Rightarrow -\infty + x = -\infty$	Absorption
$-\infty<x<\infty \Rightarrow (x-y = x-z \;=\; y=z)$	Cancellation
$-\infty<x<\infty \Rightarrow x-x = 0$	Inverse
$x<\infty \Rightarrow \infty-x = \infty$	Absorption
$-\infty<x \Rightarrow -\infty - x = -\infty$	Absorption
$-\infty<x<\infty \Rightarrow x\times0 = 0$	Base
$-\infty<x<\infty \wedge x\neq0 \Rightarrow (x\times y = x\times z \;=\; y=z)$	Cancellation
$0<x \Rightarrow x\times\infty = \infty$	Absorption
$0<x \Rightarrow x \times -\infty = -\infty$	Absorption
$-\infty<x<\infty \wedge x\neq0 \Rightarrow 0/x = 0$	Base
$-\infty<x<\infty \wedge x\neq0 \Rightarrow x/x = 1$	Base
$-\infty<y<\infty \wedge y\neq0 \Rightarrow (x/y)\times y = x$	Multiplication-Division
$-\infty<x<\infty \Rightarrow x/\infty = 0 = x/-\infty$	Annihilation
$-\infty<x<\infty \Rightarrow x^0 = 1$	Base
$-\infty<x<\infty \Rightarrow (x+y < x+z \;=\; y<z)$	Cancellation, Translation
$0<x<\infty \Rightarrow (x\times y < x\times z \;=\; y<z)$	Cancellation, Scale
$-\infty \le x \le \infty$	Extremes