345 (drunk) A drunkard is trying to walk home. At each time unit, the drunkard may go forward one distance unit, stay in the same position, or go back one distance unit. After n time units, where is the drunkard?

(a) At each time unit, there is $2/3$ probability of going forward, and $1/3$ probability of staying in the same position. The drunkard does not go back.

(b) At each time unit, there is $1/4$ probability of going forward, $1/2$ probability of staying in the same position, and $1/4$ probability of going back.

(c) At each time unit, there is $1/2$ probability of going forward, $1/4$ probability of staying in the same position, and $1/4$ probability of going back.

After trying the question, scroll down to the solution.

(a) At each time unit, there is a $2/3$ probability of going forward, and a $1/3$ probability of staying in the same position. The drunkard does not go back.

§ The drunkard's walk can be described by the following program: Let x and n be natural variables; x describes the drunkard's position, and n the remaining time.

$$x' \le n \quad \Leftarrow \quad x:= 0.\ \ 0 \le x'-x \le n$$
$$0 \le x'-x \le n \quad \Leftarrow \quad \textbf{if } n=0 \textbf{ then } ok \textbf{ else } \textbf{if } rand\ 3 < 2 \textbf{ then } x:= x+1 \textbf{ else } ok \textbf{ fi}.\ \ n:= n-1.\ \ 0 \le x'-x \le n \textbf{ fi}$$

The first occurrence of ok can be weakened to $x'=x$, and we do so in what follows. We replace $rand\ 3$ with $r: nat \to (0,..3)$ with $r\ n$ having probability $1/3$.

$$x' \le n \quad \Leftarrow \quad x:= 0.\ \ 0 \le x'-x \le n$$
$$0 \le x'-x \le n \quad \Leftarrow \quad \textbf{if } n=0 \textbf{ then } x'=x \textbf{ else } \textbf{if } r\ n < 2 \textbf{ then } x:= x+1 \textbf{ else } ok \textbf{ fi}.\ \ n:= n-1.\ \ 0 \le x'-x \le n \textbf{ fi}$$

The first implication is proven by replacing x with 0 in $0 \le x'-x \le n$. For the second implication, in its antecedent (right side), we distribute the last line over the previous **then** and **else** parts, and write the **if**s as a disjunction of conjunctions. Thus we have three cases to prove:

$$0 \le x'-x \le n \quad \Leftarrow \quad n=0 \land x'=x$$
$$0 \le x'-x \le n \quad \Leftarrow \quad n \ne 0 \land r\ n < 2 \land (x:= x+1.\ \ n:= n-1.\ \ 0 \le x'-x \le n)$$
$$0 \le x'-x \le n \quad \Leftarrow \quad n \ne 0 \land r\ n \ge 2 \land (ok.\ \ n:= n-1.\ \ 0 \le x'-x \le n)$$

The first case is easy. In the middle case, start with the antecedent:

$n \ne 0 \land r\ n < 2 \land (x:= x+1.\ \ n:= n-1.\ \ 0 \le x'-x \le n)$ — two uses of substitution

$=\ n \ne 0 \land r\ n < 2 \land 0 \le x'-(x+1) \le n-1$ — simplify

$=\ n \ne 0 \land r\ n < 2 \land 1 \le x'-x \le n$ — specialize, weaken

$\Rightarrow\ 0 \le x'-x \le n$

In the last case, start with the antecedent:

$n \ne 0 \land r\ n \ge 2 \land (ok.\ \ n:= n-1.\ \ 0 \le x'-x \le n)$ — substitution and identity

$=\ n \ne 0 \land r\ n \ge 2 \land 0 \le x'-x \le n-1$ — specialize, weaken

$\Rightarrow\ 0 \le x'-x \le n$

So far we have proven that the computation satisfies $x' \le n$. A much better answer is the probability distribution of x', which will depend on n. Let $n!$ (n factorial) be defined in the usual way:

$$n! = \Pi i: 0,..n \cdot i+1 = 1 \times 2 \times 3 \times ... \times n$$

Of the n attempts to step forward, the drunkard succeeds x' times. Out of n attempts there are $n!/(x'! \times (n-x')!)$ ways to choose x' attempts that succeed, each way having probability $(2/3)^{x'} \times (1/3)^{n-x'}$. So we hypothesize that after n attempts, the drunkard walks x' steps forward with probability $n!/(x'! \times (n-x')!) \times 2^{x'}/3^n$. That is the probability for specification $x' \le n$; the probability for the other specification $0 \le x'-x \le n$ is

$$n!/((x'-x)! \times (n-x'+x)!) \times 2^{x'-x}/3^n$$

Here's the proof: start with the implementation (right side) of the first refinement.

$x:= 0.\ \ (0 \le x'-x \le n) \times n!/((x'-x)! \times (n-x'+x)!) \times 2^{x'-x}/3^n$ — substitution

$=\ (x' \le n) \times n!/(x'! \times (n-x')!) \times 2^{x'}/3^n$

which is the specification (left side) of the first refinement. Now the last refinement, starting with its implementation, replacing $rand\ 3 < 2$ with $2/3$.

$$\textbf{if } n=0 \textbf{ then } x'=x \textbf{ else } \textbf{if } 2/3 \textbf{ then } x:= x+1 \textbf{ else } ok \textbf{ fi}.\ \ n:= n-1.\ \ (0 \le x'-x \le n) \times n!/((x'-x)! \times (n-x'+x)!) \times 2^{x'-x}/3^n \textbf{ fi}$$

distribute the line beginning $n:= n-1$ into the **then** and **else** parts of the previous line

$$= \textbf{if } n=0 \textbf{ then } x'=x \textbf{ else if } 2/3 \textbf{ then } (x:= x+1.\ \ n:= n-1.\ \ (0 \le x'-x \le n) \times n!/((x'-x)! \times (n-x'+x)!) \times 2^{x'-x}/3^n) \textbf{ else } ok.\ \ n:= n-1.\ \ (0 \le x'-x \le n) \times n!/((x'-x)! \times (n-x'+x)!) \times 2^{x'-x}/3^n \textbf{ fi fi}$$

substitutions, and ok is the identity for .

$=\quad$ **if** $n{=}0$ **then** $x'{=}x$
else if $2/3$ **then** $(0{\le}x'{-}x{-}1{\le}n{-}1) \times (n{-}1)!/((x'{-}x{-}1)!{\times}(n{-}1{-}x'{+}x{+}1)!) \times 2^{x'-x-1}/3^{n-1}$
else $(0{\le}x'{-}x{\le}n{-}1) \times (n{-}1)!/((x'{-}x)!{\times}(n{-}1{-}x'{+}x)!) \times 2^{x'-x}/3^{n-1}$ **fi fi**

let $m = x'{-}x$

$=\quad$ **if** $n{=}0$ **then** $m{=}0$
else if $2/3$ **then** $(0{\le}m{-}1{\le}n{-}1) \times (n{-}1)!/((m{-}1)!{\times}(n{-}1{-}m{+}1)!) \times 2^{m-1}/3^{n-1}$
else $(0{\le}m{\le}n{-}1) \times (n{-}1)!/(m!{\times}(n{-}1{-}m)!) \times 2^{m}/3^{n-1}$ **fi fi**

On top line use context $n{=}m{=}0$; on other lines do a numeric rearrangement.

$=\quad$ **if** $n{=}0$ **then** $(0{\le}m{\le}n) \times n!/(m!{\times}(n{-}m)!) \times 2^m/3^n$
else if $2/3$ **then** $(0{<}m) \times 3/2 \times m/n \times (0{\le}m{\le}n) \times n!/(m!{\times}(n{-}m)!) \times 2^m/3^n$
else $(m{<}n) \times 3 \times (n{-}m)/n \times (0{\le}m{\le}n) \times n!/(m!{\times}(n{-}m)!) \times 2^m/3^n$ **fi fi**

factor out common part

$=\quad$ (**if** $n{=}0$ **then** 1
else if $2/3$ **then** $(0{<}m) \times 3/2 \times m/n$ — $0{<}m$ is unnecessary due to m/n
else $(m{<}n) \times 3 \times (n{-}m)/n$ **fi fi**) — $m{<}n$ is unnecessary due to $(n{-}m)/n$
$\times\ (0{\le}m{\le}n) \times n!/(m!{\times}(n{-}m)!) \times 2^m/3^n$

$=\quad$ (**if** $n{=}0$ **then** 1
else if $2/3$ **then** $3/2 \times m/n$
else $3 \times (n{-}m)/n$ **fi fi**)
$\times\ (0{\le}m{\le}n) \times n!/(m!{\times}(n{-}m)!) \times 2^m/3^n$

use numeric definition of **if**

$$= \quad (\ (n{=}0) \times 1 + (n{>}0) \times (\ 2/3 \times 3/2 \times m/n + 1/3 \times 3 \times (n{-}m)/n\ )\ ) \times n!/(m!{\times}(n{-}m)!) \times 2^m/3^n$$

$$= \quad (\ (n{=}0) \times 1\ +\ (n{>}0) \times 1\ ) \times (0{\le}m{\le}n) \times n!/(m!{\times}(n{-}m)!) \times 2^m/3^n$$

$$= \quad (0{\le}m{\le}n) \times n!/(m!{\times}(n{-}m)!) \times 2^m/3^n$$

$$= \quad (0{\le}x'{-}x{\le}n) \times n!/((x'{-}x)!{\times}(n{-}x'{+}x)!) \times 2^{x'-x}/3^n$$

which is the specification of the second refinement. That concludes the proof. The average value of x' is

$$(x'{\le}n) \times n!/(x'!{\times}(n{-}x')!) \times 2^{x'}/3^n.\ x$$

$$= \quad \Sigma x''\cdot\ (x''{\le}n) \times n!/(x''!{\times}(n{-}x'')!) \times 2^{x''}/3^n \times x''$$

$$= \quad 2/3 \times n$$

(b) At each time unit, there is a $1/4$ probability of going forward, a $1/2$ probability of staying in the same position, and a $1/4$ probability of going back.

no solution given

(c) At each time unit, there is a $1/2$ probability of going forward, a $1/4$ probability of staying in the same position, and a $1/4$ probability of going back.

no solution given