359 (dice room) There are infinitely many people, and a madman. The madman kidnaps a person and puts the person in a room. The madman then throws a pair of six-sided dice. If the dice land as a pair of 1s, the madman murders the person. If the dice do not land as a pair of 1s, the madman releases the person, then kidnaps 10 new people. Again the madman throws the dice. If the dice land as a pair of 1s, the madman murders everyone in the room. If the dice do not land as a pair of 1s, the madman releases everyone, then kidnaps 100 new people. The madman keeps doing this, increasing the number of kidnapped people by 10 times each round, until the dice land as a pair of 1s, at which point he murders everyone in the room, and is finished.

(a) Each round, what is the probability that the people in the room are murdered?

(b) What is the average number of rounds?

(c) What is the average number of people kidnapped?

(d) What is the average number of people murdered?

(e) What is the average fraction of people kidnapped who are murdered?

After trying the question, scroll down to the solution.

(a) Each round, what is the probability that the people in the room are murdered?

§ 1 / 36

(b) What is the average number of rounds?

§ To answer this and the rest of the questions, we need to write the program. Let t: *nat* be the time variable, measured by counting rounds. Let n: 10^{nat} be the number of people kidnapped each round. Let s: *nat* count the number of people kidnapped in total. The program is

$t:= 0$. $n:= 1$. $s:= 1$. **while** 35/36 **do** $t:= t+1$. $n:= n\times 10$. $s:= s+n$ **od**

Initially the time is 0 . Initially, one person is kidnapped. Initially, that's a total of one person kidnapped. Now the dice are thrown, and if it's not a pair of 1's, we have one more round, with 10 times as many people, and that many people are added to the total, and repeat until we get a pair of 1s. I conjecture that this program is the distribution

$$(n'=10^{t'}) \times (s' = (10^{t'+1} - 1)/9) \times (35/36)^{t'} / 36$$

To prove it, I need a hypothesis for the loop alone. It is modeled on the distribution for the whole program. It is

$$(t'\ge t) \times (n' = n \times 10^{t'-t}) \times (s' = s + n\times 10\times(10^{t'-t} - 1)/9) \times (35/36)^{t'-t} / 36$$

Here is the proof of the loop hypothesis.

if 35/36
then $t:= t+1$. $n:= n\times 10$. $s:= s+n$.
$(t'\ge t) \times (n' = n \times 10^{t'-t}) \times (s' = s + n\times 10\times(10^{t'-t} - 1)/9) \times (35/36)^{t'-t} / 36$
else *ok* **fi** In **then**-part use substitution law three times.

$=$ **if** 35/36
then $(t'\ge t+1) \times (n' = n\times 10\times 10^{t'-(t+1)}) \times (s' = s + n\times 10 + n\times 10\times 10\times(10^{t'-(t+1)} - 1)/9)$
$\times (35/36)^{t'-(t+1)} / 36$
else *ok* **fi** simplify **then**-part; replace *ok*

$=$ **if** 35/36
then $(t'\ge t+1) \times (n' = n\times 10^{t'-t}) \times (s' = s + n\times 10\times(10^{t'-t} - 1)/9) \times (35/36)^{t'-(t+1)} / 36$
else $(t'=t) \times (n'=n) \times (s'=s)$ **fi** replace **if**

$=$ $35/36 \times (t'\ge t+1) \times (n' = n\times 10^{t'-t}) \times (s' = s + n\times 10\times(10^{t'-t} - 1)/9)$
$\times (35/36)^{t'-(t+1)} / 36$
$+\ 1/36 \times (t'=t) \times (n'=n) \times (s'=s)$ simplify the top term
in the bottom term, use context $t'=t$ to multiply by the neutral factors $10^{t'-t}$ and $(35/36)^{t'-t}$ and to add 0 in the form of $n\times 10\times(10^{t'-t} - 1)/9$

$=$ $(t'\ge t+1) \times (n' = n \times 10^{t'-t}) \times (s'= s + n\times 10\times(10^{t'-t} - 1)/9) \times (35/36)^{t'-t} / 36$
$+ (t'=t) \times (n' = n \times 10^{t'-t}) \times (s'= s + n\times 10\times(10^{t'-t} - 1)/9) \times (35/36)^{t'-t} / 36$
combine top and bottom terms

$=$ $(t'\ge t) \times (n' = n \times 10^{t'-t}) \times (s' = s + n\times 10\times(10^{t'-t} - 1)/9) \times (35/36)^{t'-t} / 36$

and that's the loop hypothesis. Now we prove the program distribution.

$t:= 0$. $n:= 1$. $s:= 1$.
$(t'\ge t) \times (n' = n\times 10^{t'-t}) \times (s' = s + n\times 10\times(10^{t'-t} - 1)/9) \times (35/36)^{t'-t} / 36$
substitution law three times

$=$ $(t'\ge 0) \times (n' = 1\times 10^{t'-0}) \times (s' = 1 + 1\times 10\times(10^{t'-0} - 1)/9) \times (35/36)^{t'-0} / 36$ simplify

$=$ $(n'=10^{t'}) \times (s' = (10^{t'+1} - 1)/9) \times (35/36)^{t'} / 36$

What is the average number of rounds?

$(n'=10^{t'}) \times (s' = (10^{t'+1} - 1)/9) \times (35/36)^{t'} / 36.\ t$

$= \Sigma t'', n'', s''\cdot\ (n''=10^{t''}) \times (s'' = (10^{t''+1} - 1)/9) \times (35/36)^{t''} / 36 \times t''$

$= (35/36)^0 /36 \times 0 + (35/36)^1 /36 \times 1 + (35/36)^2 /36 \times 2 + (35/36)^3 /36 \times 3 + ...$

$= 35$

(c) What is the average number of people kidnapped?

§ $(n'=10^{t'}) \times (s' = (10^{t'+1} - 1)/9) \times (35/36)^{t'} / 36.\ s$

$= \Sigma t'', n'', s''\cdot\ (n''=10^{t''}) \times (s'' = (10^{t''+1} - 1)/9) \times (35/36)^{t''} / 36 \times s''$

$= ((35/36)^0 \times (10^1 - 1) + (35/36)^1 \times (10^2 - 1)$
$+ (35/36)^2 \times (10^3 - 1) + (35/36)^3 \times (10^4 - 1) + ...) / 324$

$= \infty$

(d) What is the average number of people murdered?

§ $(n'=10^{t'}) \times (s' = (10^{t'+1} - 1)/9) \times (35/36)^{t'} / 36.\ n$

$= \Sigma t'', n'', s''\cdot\ (n''=10^{t''}) \times (s'' = (10^{t''+1} - 1)/9) \times (35/36)^{t''} / 36 \times n''$

$= (35/36)^0 / 36 \times 10^0 + (35/36)^1 / 36 \times 10^1 + (35/36)^2 / 36 \times 10^2 + ...$

$= \infty$

(e) What is the average fraction of people kidnapped who are murdered?

§ The fraction of people kidnapped who are murdered is n'/s' , which is $10^{t'}\times 9/(10^{t'+1}-1)$, which is just over $9/10$, and approaches $9/10$ as t' increases.