372 Subsection 6.0.0 gives four predicate versions of *nat* induction. Prove that they are equivalent.

After trying the question, scroll down to the solution.

§ The four predicate versions of *nat* induction are:

(a) $P\,0 \land \forall n{:}\ nat\cdot P\,n \Rightarrow P\,(n{+}1) \implies \forall n{:}\ nat\cdot P\,n$

(b) $P\,0 \lor \exists n{:}\ nat\cdot \neg P\,n \land P\,(n{+}1) \impliedby \exists n{:}\ nat\cdot P\,n$

(c) $\forall n{:}\ nat\cdot P\,n \Rightarrow P\,(n{+}1) \implies \forall n{:}\ nat\cdot P\,0 \Rightarrow P\,n$

(d) $\exists n{:}\ nat\cdot \neg P\,n \land P\,(n{+}1) \impliedby \exists n{:}\ nat\cdot \neg P\,0 \land P\,n$

Proof that (a) = (b), starting with (a).

$(P\,0 \land \forall n{:}\ nat\cdot P\,n \Rightarrow P\,(n{+}1) \implies \forall n{:}\ nat\cdot P\,n)$ — contrapositive

$= \neg(P\,0 \land \forall n{:}\ nat\cdot P\,n \Rightarrow P\,(n{+}1)) \Leftarrow \neg\forall n{:}\ nat\cdot P\,n$ — duality

$= \neg P\,0 \lor \neg(\forall n{:}\ nat\cdot P\,n \Rightarrow P\,(n{+}1)) \Leftarrow \neg\forall n{:}\ nat\cdot P\,n$ — duality twice

$= \neg P\,0 \lor (\exists n{:}\ nat\cdot \neg(P\,n \Rightarrow P\,(n{+}1))) \Leftarrow \exists n{:}\ nat\cdot \neg P\,n$ — material implication

$= \neg P\,0 \lor (\exists n{:}\ nat\cdot \neg(\neg P\,n \lor P\,(n{+}1))) \Leftarrow \exists n{:}\ nat\cdot \neg P\,n$ — duality

$= \neg P\,0 \lor (\exists n{:}\ nat\cdot \neg\neg P\,n \land \neg P(n{+}1)) \Leftarrow \exists n{:}\ nat\cdot \neg P\,n$ — double negation

$P$ is implicitly universally quantified, so rename it with its negation

$= (P\,0 \lor \exists n{:}\ nat\cdot \neg P\,n \land P\,(n{+}1) \impliedby \exists n{:}\ nat\cdot P\,n)$

Proof that (a) = (c), starting with (a).

$(P\,0 \land \forall n{:}\ nat\cdot P\,n \Rightarrow P\,(n{+}1) \implies \forall n{:}\ nat\cdot P\,n)$ — portation

$= \forall n{:}\ nat\cdot P\,n \Rightarrow P\,(n{+}1) \Rightarrow (P\,0 \Rightarrow \forall n{:}\ nat\cdot P\,n)$ — distributive

$= (\forall n{:}\ nat\cdot P\,n \Rightarrow P\,(n{+}1) \implies \forall n{:}\ nat\cdot P\,0 \Rightarrow P\,n)$

Proof that (b) = (d) starting with (b).

$(P\,0 \lor \exists n{:}\ nat\cdot \neg P\,n \land P\,(n{+}1) \impliedby \exists n{:}\ nat\cdot P\,n)$ — double negation

$= (\neg\neg P\,0 \lor \exists n{:}\ nat\cdot \neg P\,n \land P\,(n{+}1) \impliedby \exists n{:}\ nat\cdot P\,n)$ — portation

$= (\exists n{:}\ nat\cdot \neg P\,n \land P\,(n{+}1) \impliedby \neg P\,0 \land \exists n{:}\ nat\cdot P\,n)$ — distribution

$= (\exists n{:}\ nat\cdot \neg P\,n \land P\,(n{+}1) \impliedby \exists n{:}\ nat\cdot \neg P\,0 \land P\,n)$