377 Show that we can define *nat* by fixed-point construction together with

(a) $\forall n: nat\cdot\ 0 \le n < n+1$

(b) $\exists m: nat\cdot\ \forall n: nat\cdot\ m \le n < n+1$

After trying the question, scroll down to the solution.

(a) $\forall n: nat\cdot\ 0 \le n < n+1$

§ We are given

$nat = 0, nat+1$

$\forall n: nat\cdot\ 0 \le n < n+1$

and we must prove

$B = 0, B+1 \ \Rightarrow\ nat: B$

My first move is portation: given

$nat = 0, nat+1$

$\forall n: nat\cdot\ 0 \le n < n+1$

$B = 0, B+1$

prove

$nat: B$

Now I am stuck; I have no idea how to proceed. So instead of a formal proof, I offer an informal proof. Consider bunch nat to be unknown. From nat fixed-point construction we can prove that 0, 1, 2, and so on, are in nat . But maybe more elements are in nat . Maybe nat is all the integers

$..., -3, -2, -1, 0, 1, 2, 3, ...$

The integers satisfy the nat fixed-point construction axiom. But from the other given formula we know $0 \le n$ for each element n , so that rules out all the negative numbers. Maybe nat includes ∞ .

$0, 1, 2, 3, ..., \infty$

That satisfies the nat fixed-point construction axiom. But from the other given formula we know $n < n+1$ for each element n , so that rules out ∞ . Maybe there are elements in between the natural numbers, like this:

$0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, ...$

This bunch satisfies ordinary nat construction, but not fixed-point construction. If this were nat , then $0, nat+1$ would be

$0, 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, ...$

missing 0.5 . Similarly for any other elements in between the natural numbers. So nat must be the natural numbers. Most mathematicians consider this to be a proof, but I don't. I want formal proof, but I don't know how to prove it formally.

(b) $\exists m: nat\cdot \forall n: nat\cdot\ m \le n < n+1$

§ This is like part (a), but instead of $0 \le n$ we have $m \le n$ for some m . This makes it harder to rule out the negative integers.