38 Let $\bullet$ be a two-operand infix operator (precedence 3) whose operands and result are of some type T. Let $\lozenge$ be a two-operand infix operator (precedence 7) whose operands are of type T and whose result is binary, defined by the axiom

$$a \lozenge b \;=\; a \bullet b = a$$

(a) Prove if $\bullet$ is idempotent then $\lozenge$ is reflexive.
(b) Prove if $\bullet$ is associative then $\lozenge$ is transitive.
(c) Prove if $\bullet$ is symmetric then $\lozenge$ is antisymmetric.
(d) If T is the binary values and $\bullet$ is $\wedge$, what is $\lozenge$?
(e) If T is the binary values and $\bullet$ is $\vee$, what is $\lozenge$?
(f) If T is the natural numbers and $\lozenge$ is $\le$, what is $\bullet$?
(g) The axiom defines $\lozenge$ in terms of $\bullet$. Can it be inverted, so that $\bullet$ is defined in terms of $\lozenge$?

After trying the question, scroll down to the solution.

(a) Prove if $\bullet$ is idempotent then $\Diamond$ is reflexive.

§ $\quad a \bullet a = a$ — use axiom

$=\quad a \Diamond a$

(b) Prove if $\bullet$ is associative then $\Diamond$ is transitive.

§ $\quad a \Diamond b \ \land\ b \Diamond c$ — use axiom 2 times

$=\quad a \bullet b = a \ \land\ b \bullet c = b$ — idempotence of $\land$

$=\quad a \bullet b = a \ \land\ a \bullet b = a \ \land\ b \bullet c = b$ — use third conjunct to replace b in second

$=\quad a \bullet b = a \ \land\ a \bullet (b \bullet c) = a \ \land\ b \bullet c = b$ — specialize: drop third conjunct

$\Rightarrow\quad a \bullet b = a \ \land\ a \bullet (b \bullet c) = a$ — use associativity

$=\quad a \bullet b = a \ \land\ (a \bullet b) \bullet c = a$ — use first conjunct to replace $a \bullet b$ in second

$=\quad a \bullet b = a \ \land\ a \bullet c = a$ — specialize: drop first conjunct

$\Rightarrow\quad a \bullet c = a$ — use axiom

$=\quad a \Diamond c$

(c) Prove if $\bullet$ is symmetric then $\Diamond$ is antisymmetric.

§ $\quad a \Diamond b \ \land\ b \Diamond a$ — use axiom 2 times

$=\quad a \bullet b = a \ \land\ b \bullet a = b$ — use symmetry of $=$ and $\bullet$

$=\quad a = a \bullet b \ \land\ a \bullet b = b$ — transitivity of $=$

$\Rightarrow\quad a = b$

(d) If T is the binary values and $\bullet$ is $\land$, what is $\Diamond$?

§ $\quad a \Rightarrow b \ = \ a \land b = a$ so $\Diamond$ is $\Rightarrow$.

(e) If T is the binary values and $\bullet$ is $\lor$, what is $\Diamond$?

§ $\quad a \Leftarrow b \ = \ a \lor b = a$ so $\Diamond$ is $\Leftarrow$.

(f) If T is the natural numbers and $\Diamond$ is $\le$, what is $\bullet$?

§ $\quad a \le b \ = \ min\ a\ b = a$ so $\bullet$ is min .

(g) The axiom defines $\Diamond$ in terms of $\bullet$. Can it be inverted, so that $\bullet$ is defined in terms of $\Diamond$?

§ If T is the binary values we can invert as follows: $a \bullet b \ = \ a \Diamond b = a$. If T is anything else, we can invert under the assumption $a \Diamond b \ \lor\ b \Diamond a$. The inversion is

$a \bullet b \ =$ **if** $a \Diamond b$ **then** a **else** b **fi**