392 Here is a pair of mutually recursive equations.

$$even = 0, odd+1$$
$$odd = even+1$$

(a) What does recursive construction yield? Show the construction.

(b) Are further axioms needed to ensure that *even* consists of only the even naturals, and *odd* consists of only the odd naturals? If so, what axioms?

After trying the question, scroll down to the solution.

(a) What does recursive construction yield? Show the construction.

§ There are many ways to do the recursive construction. Here's one.

$$\begin{array}{lll@{\qquad}lll}
even_0 &=& null & odd_0 &=& null\\
even_1 &=& 0,\ odd_0{+}1 & odd_1 &=& even_0{+}1\\
&=& 0,\ null{+}1 & &=& null{+}1\\
&=& 0 & &=& null\\
even_2 &=& 0,\ odd_1{+}1 & odd_2 &=& even_1{+}1\\
&=& 0,\ null{+}1 & &=& 0{+}1\\
&=& 0 & &=& 1\\
even_3 &=& 0,\ odd_2{+}1 & odd_3 &=& even_2{+}1\\
&=& 0,\ 1{+}1 & &=& 0{+}1\\
&=& 0,\ 2 & &=& 1\\
even_4 &=& 0,\ odd_3{+}1 & odd_4 &=& even_3{+}1\\
&=& 0,\ 1{+}1 & &=& (0,\ 2){+}1\\
&=& 0,\ 2 & &=& 1,\ 3\\
even_5 &=& 0,\ odd_4{+}1 & odd_5 &=& even_4{+}1\\
&=& 0,\ (1,\ 3){+}1 & &=& (0,\ 2){+}1\\
&=& 0,\ 2,\ 4 & &=& 1,\ 3\\
even_{2n} &=& 2\times(0,..n) & odd_{2n} &=& 2\times(0,..n)+1\\
even_{2n+1} &=& 2\times(0,..n{+}1) & odd_{2n+1} &=& 2\times(0,..n)+1\\
even_\infty &=& 2\times nat & odd_\infty &=& 2\times nat+1
\end{array}$$

These do satisfy the equations. Another way is to substitute one of the equations into the other, forming a direct recursion. Another way is to find $even_0$, odd_1 , $even_2$, odd_3 , and so on.

(b) Are further axioms needed to ensure that *even* consists of only the even naturals, and *odd* consists of only the odd naturals? If so, what axioms?

§ Yes, further axioms are needed. Without them, *even* and *odd* could be the even and odd integers, or they could both be all the integers, or they could both be the rationals, and many other possibilities. To make them the even and odd naturals, we could add the fixed-point induction axiom

$$(B = 0,\ C{+}1) \land (C = B{+}1) \;\Rightarrow\; even{:}\ B \;\land\; odd{:}\ C$$

Or, we could add the axiom

$$\forall e{:}\ even\cdot\ 0 \le e$$

or the axiom

$$even{:}\ nat$$