47 Let $\otimes$ be a two-operand infix operator (precedence 3) with natural operands and an extended natural result. Informally, $n \otimes m$ means "the number of times that n is a factor of m". It is defined by the following two axioms.

$$m: n \times nat \ \lor \ n \otimes m = 0$$

$$n \neq 0 \ \Rightarrow \ n \otimes (m \times n) \ = \ n \otimes m + 1$$

(a) Make a 3×3 chart of the values of $(0,..3) \otimes (0,..3)$.

(b) Show that the axioms become inconsistent if the antecedent of the second axiom is removed.

(c) How should we change the axioms to allow $\otimes$ to have extended natural operands?

After trying the question, scroll down to the solution.

(a) Make a 3×3 chart of the values of $(0,..3)\otimes(0,..3)$.

§

	0	1	2
0		0	0
1	∞	∞	∞
2	∞	0	1

(b) Show that the axioms become inconsistent if the antecedent of the second axiom is removed.

§ $0\otimes0 \;=\; 0\otimes(1\times0) \;=\; 0\otimes1 + 1 \;=\; 0+1 \;=\; 1$

$0\otimes0 \;=\; 0\otimes(0\times0) \;=\; 0\otimes0 + 1$

Hence $1 \;=\; 1+1$.

(c) How should we change the axioms to allow $\otimes$ to have extended natural operands?

§ From the first axiom, instantiating with $m=\infty$ and $n=1$, we get

$\infty\!: 1\times nat \;\lor\; 1\otimes\infty = 0$

$=$ $\bot \;\lor\; 1\otimes\infty = 0$

$=$ $1\otimes\infty = 0$

From the second axiom, instantiating with $m=\infty$ and $n=1$, we get

$1\neq0 \;\Rightarrow\; 1\otimes(\infty\times1) \;=\; 1\otimes\infty + 1$

$=$ $1\otimes\infty \;=\; 1\otimes\infty + 1$ now use what we got from the first axiom

$=$ $0 = 0+1$

So we can't leave the axioms as they are. We can change *nat* to *xnat* in the first axiom; now for $n\neq0$ we have $n\otimes\infty = \infty$. Perhaps we don't want $\infty\otimes\infty = \infty$, so perhaps we should weaken the second axioms to $0<n<\infty \;\Rightarrow\; n\otimes(m\times n) \;=\; n\otimes m + 1$. We now have no answer for $\infty\otimes m$, and I don't know what it should be.