

68 (order; research) There is an ordering on extended real numbers. Using that, we defined an ordering on strings of extended real numbers. Using that, we defined an ordering on lists of extended real numbers. This exercise is to explore an extension that connects these orders, and extends them to sets. Let A be any bunch (anything), and let S be any string. Add the axioms

$$A < \{A\}$$

$$S < [S]$$

so that an increase in structure is an increase in the order. What correspondence can you make between this order and Cantor's order $\not\subset A < \not\subset A$ in the higher cardinals?

After trying the question, scroll down to the solution.

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Here is a line in the order we already had before adding the axioms.

$$0 < 1 < 2 < \dots < \infty < \infty; 0 < \infty; 1 < \infty; 2 < \dots < \infty; \infty < \infty; \infty; 0 < \infty; \infty; 1 < \dots < \infty * \infty$$

That line had a last element. There were already many other lines. For example,

$$0 < 0; 0 < 0; 0; 0 < \dots < \infty * 0$$

Any such line of strings gave rise to a line of lists. For example,

$$[0] < [0;0] < [0;0;0] < \dots < [\infty * 0]$$

But nothing related a line of strings to a line of lists. With the new axioms, we now have

$$0 < [0] < [[0]] \text{ and } 0;0 < [0;0] < [[0;0]] \text{ and so on. Also } 0 < \{0\} < \{\{0\}\} \text{ and } 0,1 < \{0,1\} < \{\{0,1\}\} \text{ and so on. Lots and lots of order.}$$

I don't know enough about higher cardinalities to answer the question. As far as I know, higher cardinalities have no applications (outside mathematics), so they are defined by arbitrary, unmotivated axioms. So I have no motivation to learn about them. (See [the Size of a Set.](#))