80 (whodunit) Here are ten statements.

(i) Some criminal robbed the Russell mansion.

(ii) Whoever robbed the Russell mansion either had an accomplice among the servants or had to break in.

(iii) To break in one would have to either smash the door or pick the lock.

(iv) Only an expert locksmith could pick the lock.

(v) Anyone smashing the door would have been heard.

(vi) Nobody was heard.

(vii) No one could rob the Russell mansion without fooling the guard.

(viii) To fool the guard one must be a convincing actor.

(ix) No criminal could be both an expert locksmith and a convincing actor.

Therefore

(x) Some criminal had an accomplice among the servants.

(a) Choosing good abbreviations, translate each of these statements into formal logic.

(b) Taking the first nine statements as axioms, prove the last statement.

After trying the question, scroll down to the solution.

(a) Choosing good abbreviations, translate each of these statements into formal logic.

§ Here are some abbreviations.

$C\,x = (\,x$ is a criminal$)$
$R\,x = (\,x$ robbed the Russell mansion$)$
$S\,x = (\,x$ had an accomplice among the servants$)$
$B\,x = (\,x$ broke in$)$
$D\,x = (\,x$ smashed the door$)$
$P\,x = (\,x$ picked the lock$)$
$L\,x = (\,x$ is an expert locksmith$)$
$H\,x = (\,x$ was heard$)$
$F\,x = (\,x$ fooled the guard$)$
$A\,x = (\,x$ is a convincing actor$)$

Now the statements are formalized as follows.

(i) $\exists x\cdot\ C\,x \wedge R\,x$
(ii) $\forall x\cdot\ R\,x \Rightarrow S\,x \vee B\,x$
(iii) $\forall x\cdot\ B\,x \Rightarrow D\,x \vee P\,x$
(iv) $\forall x\cdot\ L\,x \Leftarrow P\,x$
(v) $\forall x\cdot\ D\,x \Rightarrow H\,x$
(vi) $\neg\exists x\cdot\ H\,x$
(vii) $\neg\exists x\cdot\ R\,x \wedge \neg\, F\,x$
(viii) $\forall x\cdot\ F\,x \Rightarrow A\,x$
(ix) $\neg\exists x\cdot\ C\,x \wedge L\,x \wedge A\,x$

Therefore

(x) $\exists x\cdot\ C\,x \wedge S\,x$

(b) Taking the first nine statements as axioms, prove the last statement.

§ Lemma:

	$\top$	(vii)
$=$	$\neg\exists x\cdot\ R\,x \wedge \neg\, F\,x$	duality (deMorgan)
$=$	$\forall x\cdot\ \neg(R\,x \wedge \neg\, F\,x)$	duality (deMorgan)
$=$	$\forall x\cdot\ \neg R\,x \vee \neg\neg\, F\,x$	double negation
$=$	$\forall x\cdot\ \neg\, R\,x \vee F\,x$	material implication
$=$	$\forall x\cdot\ R\,x \Rightarrow F\,x$	

Now the main proof:

	$\top$	(i)
$=$	$\exists x\cdot\ C\,x \wedge R\,x$	idempotence
$=$	$\exists x\cdot\ C\,x \wedge R\,x \wedge R\,x$	lemma and (ii)
$\Longrightarrow$	$\exists x\cdot\ C\,x \wedge F\,x \wedge (S\,x \vee B\,x)$	(viii) and (iii)
$\Longrightarrow$	$\exists x\cdot\ C\,x \wedge A\,x \wedge (S\,x \vee D\,x \vee P\,x)$	(v) and (iv)
$\Longrightarrow$	$\exists x\cdot\ C\,x \wedge A\,x \wedge (S\,x \vee H\,x \vee L\,x)$	distribute
$=$	$\exists x\cdot\ C\,x \wedge A\,x \wedge S\,x\ \vee\ C\,x \wedge A\,x \wedge H\,x\ \vee\ C\,x \wedge A\,x \wedge L\,x$	(vi) and (ix)
$=$	$\exists x\cdot\ C\,x \wedge A\,x \wedge S\,x$	specialize
$\Longrightarrow$	$\exists x\cdot\ C\,x \wedge S\,x$	