84 There are four binary two-operand associative symmetric operators with an identity. We used two of them to define quantifiers. What happened to the other two?

After trying the question, scroll down to the solution.

§ The four binary two-operand associative symmetric operators with an identity are $\wedge$ $\vee$ $=$ $\ne$. Since $=$ and $\ne$ are not idempotent, we should define their quantifiers $\mathcal{E}$ and $\mathcal{O}$ as follows.

$$\mathcal{E}v\text{: } null\cdot\ b\ \ =\ \ \top$$
$$\mathcal{E}v\text{: } x\cdot\ b\ \ =\ \ \langle v\text{: } x\cdot\ b\rangle\ x$$
$$(\mathcal{E}v\text{: } A, B\cdot\ b) = (\mathcal{E}v\text{: } A\text{‘}B\cdot\ b)\ \ =\ \ (\mathcal{E}v\text{: } A\cdot\ b) = (\mathcal{E}v\text{: } B\cdot\ b)$$
$$\mathcal{E}v\text{: } (\S v\text{: } D\cdot\ b)\cdot\ c\ \ =\ \ \mathcal{E}v\text{: } D\cdot\ b \Rightarrow c$$

$$\mathcal{O}v\text{: } null\cdot\ b\ \ =\ \ \bot$$
$$\mathcal{O}v\text{: } x\cdot\ b\ \ =\ \ \langle v\text{: } x\cdot\ b\rangle\ x$$
$$(\mathcal{O}v\text{: } A, B\cdot\ b) \ne (\mathcal{O}v\text{: } A\text{‘}B\cdot\ b)\ \ =\ \ (\mathcal{O}v\text{: } A\cdot\ b) \ne (\mathcal{O}v\text{: } B\cdot\ b)$$
$$\mathcal{O}v\text{: } (\S v\text{: } D\cdot\ b)\cdot\ c\ \ =\ \ \mathcal{O}v\text{: } D\cdot\ b \wedge c$$

According to these definitions, $\mathcal{E}p$ says “there is an even number of elements in the domain of p for which the result is $\bot$ ”, and $\mathcal{O}p$ says “there is an odd number of elements in the domain of p for which the result is $\top$ ”. These quantifiers are known as “parity” functions, and are useful for memory error detection and correction circuits.