97 Express formally that

(a) natural n is the largest proper (neither 1 nor m) factor of natural m .

(b) g is the greatest common divisor of naturals a and b .

(c) m is the lowest common multiple of naturals a and b .

(d) p is a prime number.

(e) n and m are relatively prime numbers.

(f) there is at least one and at most a finite number of naturals satisfying predicate p .

(g) there is no smallest integer.

(h) between every two rational numbers there is another rational number.

(i) list L is a longest segment of list M that does not contain item x .

(j) the segment of list L from (including) index i to (excluding) index j is a segment whose sum is smallest.

(k) a and b are items of lists A and B (respectively) whose absolute difference is least.

(l) p is the length of a longest plateau (segment of equal items) in a nonempty sorted list L .

(m) all items that occur in list L occur in a segment of length 10 .

(n) all items of list L are different (no two items are equal).

(o) at most one item in list L occurs more than once.

(p) the maximum item in list L occurs m times.

(q) list L is a permutation of list M .

After trying the question, scroll down to the solution.

§ This exercise illustrates the need for formalization. Even carefully worded informal specifications can be misunderstood.

(a) natural n is the largest proper (neither 1 nor m) factor of natural m .

§ Define predicate p so that $p\ x$ means that x is a proper factor of m .

$$p \;=\; \langle x\text{: } nat\cdot\ m\text{: } x\times nat \wedge x\ne 1 \wedge x\ne m\rangle$$

Then the expression we want is

$$p\ n \wedge \forall x\text{: } nat\cdot\ p\ x \Rightarrow x\le n$$

or equivalently

$$n \;=\; \Uparrow x\text{: } (\S p)\cdot\ x$$

(b) g is the greatest common divisor of naturals a and b .

§ $$a, b\text{: } g\times nat \;\wedge\; \neg\exists h\text{: } nat\cdot\ a, b\text{: } h\times nat \;\wedge\; h>g$$

(c) m is the lowest common multiple of naturals a and b .

§ Is 0 a multiple of a and b ? If so, the answer is $m=0$. If not, then

$$m\text{: } (nat+1)\times a \text{ ‘ } (nat+1)\times b \;\wedge\; \forall z\text{: } (nat+1)\times a \text{ ‘ } (nat+1)\times b\cdot\ m\le z$$

or $$m \;=\; \Downarrow m\text{: } (nat+1)\times a \text{ ‘ } (nat+1)\times b\cdot\ m$$

(d) p is a prime number.

§ Here are some possible answers.

$\forall n\text{: } nat\cdot\ p\text{: } n\times nat \Rightarrow n=1 \vee n=p$	says 0 isn't prime and 1 is
$\forall n\text{: } 0,..p\cdot\ p\text{: } n\times nat \Rightarrow n=1$	says 0 and 1 are prime
$¢\S n\text{: } 0,..p\cdot\ p\text{: } n\times nat \;=\; 1$	says 0 and 1 aren't prime
$¢\S n\text{: } nat\cdot\ p\text{: } n\times nat \;=\; 2$	says 0 and 1 aren't prime
$p\text{: } nat+2 \;\wedge\; \neg(p\text{: } (nat+2)\times(nat+2))$	says 0 and 1 aren't prime

These answers agree on whether numbers in $nat+2$ are prime, but they disagree on 0 and 1 . Most mathematicians want to exclude 1 , and they haven't thought about 0 .

(e) n and m are relatively prime numbers.

§ $$\forall f\text{: } nat\cdot\ n, m\text{: } f\times nat \;\Rightarrow\; f=1$$

or $$¢\S f\text{: } nat\cdot\ n, m\text{: } f\times nat \;=\; 1$$

(f) there is at least one and at most a finite number of naturals satisfying predicate p .

§ $$1 \le ¢(\S n\text{: } nat\cdot\ pn) < \infty$$

and if $\square p = nat$,

$$1 \le ¢(\S p) < \infty$$

(g) there is no smallest integer.

§ $$\neg\exists i\text{: } int\cdot\ \forall j\text{: } int\cdot\ i \le j$$

OR $$\forall i\text{: } int\cdot\ \exists j\text{: } int\cdot\ j < i$$

(h) between every two rational numbers there is another rational number.

§ $$\forall x, z\text{: } rat\cdot\ x<z \;\Rightarrow\; \exists y\text{: } rat\cdot\ x<y<z$$

One could argue that it should be

$$\forall x, z\text{: } rat\cdot\ \exists y\text{: } rat\cdot\ x<y<z \vee z<y<x$$

because it doesn't say “between every two different rational numbers”, or one could argue that “two rational numbers” means “two different rational numbers”.

(i) list L is a longest segment of list M that does not contain item x .

§ Let the type of item be T . Define

$$P \;=\; \langle A\text{: } [*T]\cdot\ \exists i, j\text{: } 0,..\#M+1\cdot\ \ i\le j \wedge A = M[i;..j] \wedge \neg\ x\text{: } M(i,..j)\rangle$$

so that $P\ A$ means " A is a segment of M that does not contain item x ". Now the answer is

$$P\ L\ \wedge\ \neg\exists A\colon [*T]\cdot\ P\ A\ \wedge\ \#A>\#L$$

(j) the segment of list L from (including) index i up to (excluding) index j is a segment whose sum is smallest.

§ $$0\le i\le j\le \#L \wedge \forall x\colon 0,..\#L+1\cdot\ \forall y\colon x,..\#L+1\cdot\ (\Sigma\ L[i;..j]) \le (\Sigma\ L[x;..y])$$

(k) a and b are items of lists A and B (respectively) whose absolute difference is least.

§ $$a\colon A(\square A)\ \wedge\ b\colon B(\square B)$$
$$\wedge\ \neg(\exists c\colon A(\square A)\cdot\ \exists d\colon B(\square B)\cdot\ abs\ (c-d) < abs\ (a-b))$$
or $$(\exists i\colon \square A\cdot\ \exists j\colon \square B\cdot\ A\ i = a\ \wedge\ B\ j = b)$$
$$\wedge\ \neg(\exists i\colon \square A\cdot\ \exists j\colon \square B\cdot\ abs\ (A\ i - B\ j) < abs\ (a-b))$$

(l) p is the length of a longest plateau (segment of equal items) in a non-empty sorted list L .

§ Define

$$P\ =\ \langle p\colon 1,..\#L+1\cdot\ \exists i\colon 0,..\#L+1-p\cdot\ L\ i = L\ (i+p-1)\rangle$$

so that $P\ p$ says that p is the length of a plateau in a non-empty sorted list L . Now the answer is $P\ p\ \wedge\ \neg\ P\ (p+1)$.

(m) all items that occur in list L occur in a segment of length 10 .

§ It doesn't say the segment must be in list L , but surely that is intended. Must an item occur as many times in the segment as in the list? Probably not. Can it be a different segment for each item? If so, it's trivially true for any list of length at least 10 . It must mean that in list L there is a segment of length 10 containing all items that occur anywhere in list L .

$$\#L\ge 10\ \wedge\ \exists i\colon 0,..\#L-9\cdot\ \forall j\colon 0,..\#L\cdot\ \exists k\colon i,..i+10\cdot\ L\ j = L\ k$$
or $$\#L\ge 10\ \wedge\ \exists i\colon 0,..\#L-9\cdot\ L(0,..\#L)\colon L(i,..i+10)$$

(n) all items of list L are different (no two items are equal).

§ $$\neg\ \exists i, j\colon \square L\cdot\ i\ne j \wedge L\ i = L\ j$$
or $$¢L(0,..\#L) = \#L$$

(o) at most one item in list L occurs more than once.

§ $$\neg\exists i, j, k, l\colon \square L\cdot\ i\ne j \wedge k\ne l \wedge L\ i = L\ j \ne L\ k = L\ l$$

(p) the maximum item in list L occurs m times.

§ If the question means exactly m times, then the answer is

$$¢§i\colon \square L\cdot\ L\ i = ⇑L\ =\ m$$

If it means at least m times, then the answer is

$$(¢§i\colon \square L\cdot\ L\ i = ⇑L)\ \ge\ m$$

(q) list L is a permutation of list M .

§ $$\forall x\cdot\ (¢§i\colon \square L\cdot\ L\ i = x)\ =\ (¢§i\colon \square M\cdot\ M\ i = x)$$