98 (friends) Formalize and prove the statement “The people you know are those known by all who know all whom you know.”.

After trying the question, scroll down to the solution.

§ I need a notation to mean “person a knows person b”; I will use $a \vdash b$ with precedence level 3. I extend $\vdash$ to bunch operands as follows.

$$A \vdash B \;=\; \forall a{:}\, A\cdot\ \forall b{:}\, B\cdot\ a \vdash b$$

Perhaps the word “you” refers to some particular person whom I will call u, or perhaps the word “you” means an arbitrary person, in which case we just put $\forall u\cdot$ in front of everything. Perhaps the word “are” means “are included among”, or perhaps it means “are exactly”; we can prove the latter, which is stronger and implies the former. All quantifications will be over people, so I won't bother to write the domains. Now let's take it slowly.

“all whom you know” $=$ $\S c\cdot\ u \vdash c$

“all who know all whom you know” $=$ $\S b\cdot\ b \vdash \S c\cdot\ u \vdash c$

“those known by all who know all whom you know” $=$ $\S a\cdot\ (\S b\cdot\ b \vdash \S c\cdot\ u \vdash c) \vdash a$

And finally, the given statement becomes

$$\S a\cdot\ u \vdash a \quad = \quad \S a\cdot\ (\S b\cdot\ b \vdash \S c\cdot\ u \vdash c) \vdash a$$

Instead of using the solution quantifier $\S$, I could have used $\forall$ according to the following three identities.

(a) $(\S x\cdot\ p) = (\S x\cdot\ q) \quad = \quad \forall x\cdot\ p = q$

(b) $(\S x\cdot\ p) \vdash y \quad = \quad \forall x\cdot\ p \Rightarrow x \vdash y$

(c) $x \vdash (\S y\cdot\ p) \quad = \quad \forall y\cdot\ p \Rightarrow x \vdash y$

So the given statement is transformed as follows.

$\quad (\S a\cdot\ u \vdash a) \;=\; (\S a\cdot\ (\S b\cdot\ b \vdash \S c\cdot\ u \vdash c) \vdash a)$ use (a)

$=\quad \forall a\cdot\ u \vdash a \;=\; (\S b\cdot\ b \vdash \S c\cdot\ u \vdash c) \vdash a$ use (b)

$=\quad \forall a\cdot\ u \vdash a \;=\; (\forall b\cdot\ (b \vdash \S c\cdot\ u \vdash c) \Rightarrow b \vdash a)$ use (c)

$=\quad \forall a\cdot\ u \vdash a \;=\; (\forall b\cdot\ (\forall c\cdot\ u \vdash c \Rightarrow b \vdash c) \Rightarrow b \vdash a)$

Now for the proof. I'll work inside the $\forall a\cdot$ and divide the proof into two cases.

if $u \vdash a$

then $(\quad (u \vdash a \;=\; (\forall b\cdot\ (\forall c\cdot\ u \vdash c \Rightarrow b \vdash c) \Rightarrow b \vdash a))$ assumption $u \vdash a$

$=\quad \forall b\cdot\ (\forall c\cdot\ u \vdash c \Rightarrow b \vdash c) \Rightarrow b \vdash a$ Specialize c to a. This weakens an antecedent, and so strengthens the implication.

$\Leftarrow\quad \forall b\cdot\ (u \vdash a \Rightarrow b \vdash a) \Rightarrow b \vdash a$ assumption $u \vdash a$

$=\quad \forall b\cdot\ b \vdash a \Rightarrow b \vdash a$ reflexive then idempotent

$=\quad \top\ )$

else $(\quad (u \vdash a \;=\; (\forall b\cdot\ (\forall c\cdot\ u \vdash c \Rightarrow b \vdash c) \Rightarrow b \vdash a))$ assumption $\neg(u \vdash a)$

$=\quad \neg \forall b\cdot\ (\forall c\cdot\ u \vdash c \Rightarrow b \vdash c) \Rightarrow b \vdash a$ Specialize b to u. This weakens a negand, and so strengthens the negation.

$\Leftarrow\quad \neg((\forall c\cdot\ u \vdash c \Rightarrow u \vdash c) \Rightarrow u \vdash a)$ reflexivity, idempotence, assumption

$=\quad \neg(\top \Rightarrow \bot)$

$=\quad \top\ )$

Here is a third approach. For bunch of people A, define $\dashv A$ to be those known by all A, and define $\models B$ to be those who know all B.

$$\dashv A \;=\; \S b\cdot\ A \vdash b$$

$$\models B \;=\; \S a\cdot\ a \vdash B$$

Then the statement we are asked to prove is $\dashv u \;=\; \dashv \models \dashv u$. Before proving it, we prove the lemma $A{:}\, \models B \;=\; B{:}\, \dashv A$ (which says that $\dashv$ and $\models$ are strongly Galois connected).

$\quad A{:}\, \models B$

$=\quad \forall a{:}\, A\cdot\ \forall b{:}\, B\cdot\ a \vdash b$

$=\quad \forall b{:}\, B\cdot\ \forall a{:}\, A\cdot\ a \vdash b$

$=\quad B{:}\, \dashv A$

Now the theorem:

$\Dashv u \;=\; \Dashv\vDash\Dashv u$

$=\quad \Dashv u\!: \Dashv\vDash\Dashv u \;\wedge\; \Dashv\vDash\Dashv u\!: \Dashv u$ use the lemma in each conjunct

$=\quad \vDash\Dashv u\!: \vDash\Dashv u \;\wedge\; u\!: \vDash\Dashv\vDash\Dashv u$ in left conjunct : is reflexive

$=\quad u\!: \vDash\Dashv\vDash\Dashv u$ transitivity

$\Leftarrow\quad u\!: \vDash\Dashv u \;\wedge\; \vDash\Dashv u\!: \vDash\Dashv\vDash\Dashv u$ use the lemma in each conjunct

$=\quad \Dashv u\!: \Dashv u \;\wedge\; \Dashv\vDash\Dashv u\!: \Dashv\vDash\Dashv u$ reflexivity twice

$=\quad \top$

The theorem is instantly generalizable to $\Dashv A = \Dashv\vDash\Dashv A$ with no change in the proof. It is further generalizable to a relation whose left and right operands come from different populations.