X4.2 Let P and Q be specifications, let I be an assertion with all nonlocal variables unprimed, and let I' be the same as I but with primes on all nonlocal variables. Prove

(a) $\qquad I{\Rightarrow}I' . I{\Rightarrow}I' \implies I{\Rightarrow}I'$

(b) $\qquad$ If I is invariant for P, and I is invariant for Q, then I is invariant for $(P.Q)$.

After trying the question, scroll down to the solution.

(a) $I{\Rightarrow}I' \,.\, I{\Rightarrow}I' \;\Longrightarrow\; I{\Rightarrow}I'$

§ $I{\Rightarrow}I' \,.\, I{\Rightarrow}I'$ rewrite with explicit arguments
   $=\quad I\,\sigma \Rightarrow I\,\sigma' \,.\, I\,\sigma \Rightarrow I\,\sigma'$ sequential composition
   $=\quad \exists\sigma''\!\cdot (I\,\sigma \Rightarrow I\,\sigma'') \wedge (I\,\sigma'' \Rightarrow I\,\sigma')$ transitivity
   $\Longrightarrow\quad \exists\sigma''\!\cdot I\,\sigma \Rightarrow I\,\sigma'$ unused quantifier
   $=\quad I\,\sigma \Rightarrow I\,\sigma'$ rewrite with implicit arguments
   $=\quad I{\Rightarrow}I'$

(b)   If I is invariant for P , and I is invariant for Q , then I is invariant for $(P.Q)$.

§   We are given I is invariant for P , which means $(I{\Rightarrow}I') \Leftarrow P$ is a theorem.
   We are given I is invariant for Q , which means $(I{\Rightarrow}I') \Leftarrow Q$ is a theorem.
   We want to prove I is invariant for $(P.Q)$, which means $(I{\Rightarrow}I') \Leftarrow (P.Q)$ is a theorem.
   One of the Refinement by Steps laws says: If $A \Leftarrow B.C$ and $B \Leftarrow D$ and $C \Leftarrow E$
   are theorems, then $A \Leftarrow D.E$ is a theorem. Let A , B , and C all be $I{\Rightarrow}I'$; let D be
   P ; let E be Q . Then we have $(A \Leftarrow B.C)$ by part (a). We have $(B \Leftarrow D)$ because
   I is invariant for P . And we have $(C \Leftarrow E)$ because I is invariant for Q . So by
   Refinement by Steps, we have $(A \Leftarrow D.E)$, which says I is invariant for $(P.Q)$.