Good Codes and Minimum Distance

Recall that for a code to be guaranteed to correct up to t errors, it's minimum distance must be at least $2t + 1$.

What's the minimum distance for the random codes used to prove the noisy coding theorem?

A random n-bit code is very likely to have minimum distance $d \le n/2$ — if we pick two codewords randomly, about half their bits will differ. So these codes are likely *not guaranteed* to correct patterns of $n/4$ or more errors.

A BSC with error probability Q will produce about nQ errors. So for $Q > 1/4$, we expect to get more errors than the code is guaranteed to correct. Yet we know these codes are good!

Conclusion: A code may be able to correct *almost all* patterns of t errors even if it can't correct *all* such patterns.

What a Good Linear Code Looks Like

Minimum distance isn't the whole story, but nevertheless, it's not good for a linear code to have very low-weight code words (and hence very small minimum distance).

A consequence: The generator matrix for a good code should not be sparse — each row should have many 1s, so that encoding a message with only one 1 produces a codeword that has many 1s.

The decoder's perspective: To be confident of decoding correctly, getting even *one* bit wrong should produce a large change in the codeword, which will be noticeable (unless we're very unlucky).

Low Density Parity Check Codes

We should avoid sparse generator matrices. But can we use a sparse parity-check matrix?

Doing so isn't *quite* optimal, but such "Low Density Parity Check" (LDPC) codes can be very good.

The big advantage of LDPC codes: There is a *computationally feasible* way of decoding them that is good, though not optimal.

We can construct LDPC codes randomly, in various ways. One way: to make an $[n, k]$ code, randomly generate columns of H with exactly three 1s in them.

For best results, equalize the number of 1s in each row (as much as possible) by randomly picking the position of the three 1s in the next column from among rows that don't already have $3n/(n-k)$ 1s in them.

Example: A $[50, 25]$ *LDPC Code*

Here's the parity-check matrix for a small LDPC code (three 1s in each column, six in each row).

```
00110000010100000000000000100000000000000000001000
00000000000000000000101000000100001001000000010000
00101010000000000001000100000100000000000000000000
00000101000000001000000000000000000000110010000000
00000000010000000001000000000001100000000001000010
00000100000010000000001000000000000100000000001010
00000000000101000100000000000000010010000010000000
01000010000100000010000000000000001000000000000001
00000000000000011010000000001000000010000100000000
01000000000010000000000011001000000000001000000000
00000000001010100000000000000000000000011000010000
00000000000000000010000000010000100000000000001101
00010000000000010000000110010000000001000000000000
10000001000000010000100000000101000000000000000000
00010010001000000000000000100000010100000000000000
00000001000001000000010000000000100000000011000000
10000000010000100000000000001000000000000000000101
00000000000000000100010001000010000010000001000000
00101000100000000100000000000000010000100000000000
10000000001000000000000000010000001001000000000100
00000000100000000000001100000010000000110000000000
00000100000000001001100000100000000000000100000000
00000000100001100000000001000000000100000000100000
01000000000000000000010010000000000000001100100000
00001000000000000000000000000011000000000000110010
```

A Generator Matrix for the Example

A systematic generator matrix obtained from the parity-check matrix (with columns re-ordered):

```
10000000000000000000000000110000011111011101001010
01000000000000000000000001001010101001101110111000
00100000000000000000000000111001100110000100111000
00010000000000000000000000000101101011111110010010
00001000000000000000000001010011000110001000100001
00000100000000000000000000111000001111010001011011
00000010000000000000000001111010111111010001011011
00000001000000000000000001011011011011100010011011
00000000100000000000000000100101000110011001000100
00000000010000000000000001001011001111010000110000
00000000001000000000000001000010101001100110100011
00000000000100000000000000010000100000000000001101
00000000000010000000000000000011000000000000110010
00000000000001000000000001010010100000100110100111
00000000000000100000000000001001100110101110000011
00000000000000010000000000111101100100011101001001
00000000000000001000000001000001001101010101010000
00000000000000000100000001010010100110001001100011
00000000000000000010000000111011100000000110001011
00000000000000000001000000110001001100110011001100
00000000000000000000100001110010000010101000101000
00000000000000000000010000000011001111010100010000
00000000000000000000001000111110100010110110000000
00000000000000000000000100110010000011101011111100
00000000000000000000000011001110100000101110101000
```

Decoding LDPC Codes

To encode a message with an LDPC, we just multiply it by the generator matrix. But how do we decode?

The optimal method (assuming a BSC, and equally-probable messages) is to pick the codeword nearest to what was received. But this is computationally infeasible.

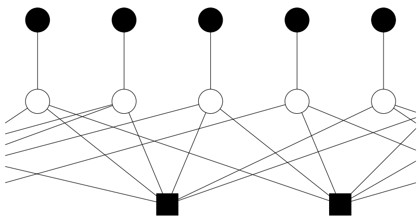
The reason LDPC codes are interesting is that the sparseness of their parity-check matrices allows for an *approximate* (good, but not optimal) decoding method that works by *propagating probabilities* through a graph.

Graphical Representation of a Code

We can represent a code by a graph:

- Empty circles represent bits of a codeword.
- Black circles represent received data bits.
- Black squares represent parity checks.

Here's a fragment of such a graph:



Notice that each codeword bit connects to three parity checks — corresponding to the three 1s in each column of H. Each parity check connects to six codeword bits.

Our task: Fill in the empty circles.

Decoding by Propagating Probabilities

We can't be absolutely sure of the codeword bits, but we can keep track of the *odds* in favour of 1 over 0 (the ratio of the probability of 1 over the probability of 0).

Each black node will send each codeword bit it connects to a message giving its idea of what the odds for 1 over 0 should be for that bit.

All the messages a codeword receives are multiplied to give the current idea of what the odds are for that bit — used to guess the codeword once these odds have stabilized.

But first, we iterate: Each codeword bit sends each parity check it connects to a message with its current odds, which the parity check node uses to update its messages to other codeword bits. Messages propagate until the odds have stabilized.

Details of the Messages

Received data bit to codeword bit: Odds sent are $(1-Q)/Q$ if the received data is 1, $Q/(1-Q)$ if the received data is 0.

Parity check to codeword bit: Message is the probability of the parity check being satisfied if that bit is 1, divided by the probability if that bit is 0. These probabilities are calculated based on that parity check's idea of the odds for the *other* bits in the parity check being 1 versus 0.

Codeword bit to a parity check: Message is the odds of the bit being 1 versus 0, based on the received data, and on the messages from the *other* parity checks the codeword bit is involved in.

Avoiding Double-Counting Information

Messages send between codeword bits and parity checks exclude information obtained from the node the message is being sent to. This avoids undesirable "double-counting" of information when a message comes back from that node.

But: This works perfectly only if the graph is a tree. If there are cycles in the graph, information can return to its source indirectly.

This is why probability propagation is only an *approximate* decoding method. It works well up to a point, but doesn't have as low an error rate as nearest-neighbor (maximum likelihood) decoding would achieve.

Demonstration of LDPC Codes

I tried rate 1/2 LDPC codes with three bits in each column of H, with varying codeword lengths, tested using a BSC with varying error probability, Q (and hence capacity, C).

Here are the block error rates for three such codes, estimated from 1000 simulated messages:

Q	C	$[100, 50]$	$[1000, 500]$	$[10000, 5000]$
0.02	0.86	0.000	0.000	0.000
0.03	0.81	0.012	0.000	0.000
0.04	0.76	0.059	0.000	0.000
0.05	0.71	0.108	0.000	0.000
0.06	0.67	0.213	0.005	0.000
0.07	0.63	0.327	0.104	0.000
0.08	0.60	0.482	0.404	0.125

Tests were done with software available from my web page, `http://www.cs.utoronto.ca/~radford/`

History of LDPC and Related Codes

- Gallager, LDPC codes — 1961.

  True merits not realized? Computers too slow? Largely ignored and forgotten.

- Berrou, *et al*, TURBO codes — 1993.

  Surprisingly good codes, practically decodable, but not really understood.

- MacKay and Neal — 1995.

  Reinvent LDPC codes, slightly improved. Show they're almost as good as TURBO codes. Decoding algorithm related to other probabilistic inference methods.

- Many (Richardson, Frey, etc.) — ongoing.

  Further improvements in LDPC codes, relationship to TURBO codes, theory of why it all works.