A Formal Theory for the Complexity Class Associated with the Stable Marriage Problem

Stephen Cook

Joint work with Dai Tri Man Lê and Yuli Ye

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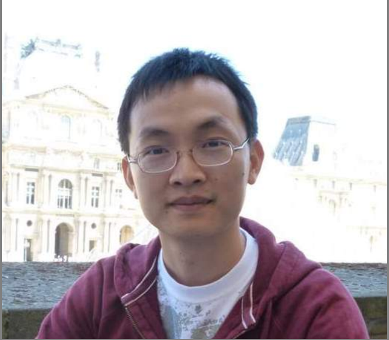
CSL 2011



Dai Lê

Bounded Reverse Mathematics [Cook-Nguyen '10]

Motivation

Classify theorems according to the computational complexity of concepts needed to prove them.

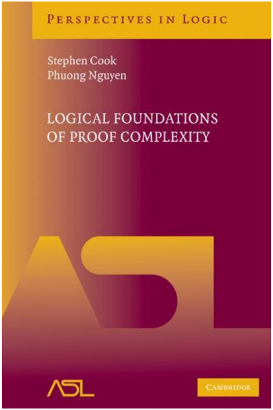
Program in Chapter 9

1. Introduce a general method for associating a canonical minimal theory VC for certain complexity classes C

$$\mathsf{AC}^0 \subseteq \mathsf{C} \subseteq \mathsf{P}$$

2. Given a theorem τ, try to find the smallest complexity class C such that

$$\mathsf{VC} \vdash \tau$$



Outline of the talk

1. The complexity class CC
    - Interesting natural complete problems: stable marriage, lex-first maximal matching, comparator circuit value problem...
2. Use the Cook-Nguyen method to define a theory for CC
3. Discuss many open problems related to CC

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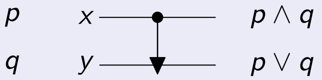
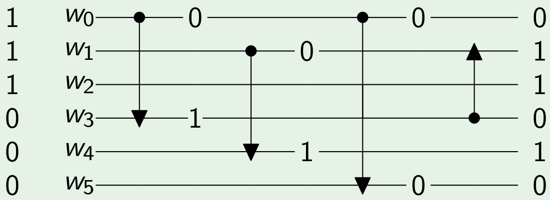
Comparator Circuits

- Originally invented for sorting, e.g.,
  - Batcher's $\mathcal{O}(\log^2 n)$-depth sorting networks ('68)
  - Ajtai-Komlós-Szemerédi (AKS) $\mathcal{O}(\log n)$-depth sorting networks ('83)
- Can also be considered as boolean circuits.

Comparator gate

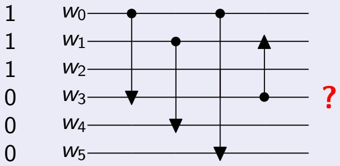


Example



Comparator Circuit Value ($\textsc{Ccv}$) Problem (decision)

Given a comparator circuit with specified Boolean inputs, determine the output value of a designated wire.



Complexity classes

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   - [Subramanian's PhD thesis '90], [Mayr-Subramanian '92]

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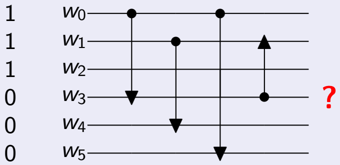

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   - Complete problems: stable marriage, lex-first maximal matching...
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   - Needed when developing a Cook-Nguyen style theory for CC
   - The function class FCC^* is closed under compostion

$$\boxed{\mathsf{NC}^1 \subseteq \mathsf{NL} \subseteq \mathsf{CC} \subseteq \mathsf{CC}^{\mathsf{Subr}} \subseteq \mathsf{CC}^* \subseteq \mathsf{P}}$$

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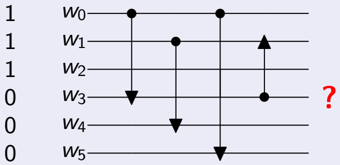

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Stable Marriage Problem (search version) (Gale-Shapley '62)

- Given n men and n women together with their preference lists
- Find a stable marriage between men and women, i.e.,
  1. a perfect matching
  2. satisfies the stability condition: no two people of the opposite sex like each other more than their current partners

Preference lists

Men:

a	x	y
b	y	x

Women:

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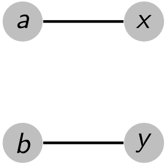

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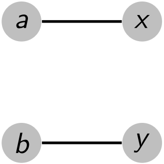

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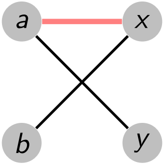

unstable marriage

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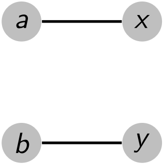

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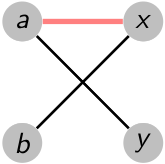


unstable marriage

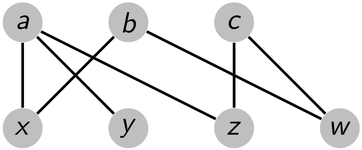
Stable Marriage Problem (decision version)

Is a given pair of (m, w) in the man-optimal (woman-optimal) stable marriage?

Lex-first maximal matching problem

Lex-first maximal matching

- Let G be a bipartite graph.
- Successively match the bottom nodes $x, y, z, \ldots$ to the least available top node



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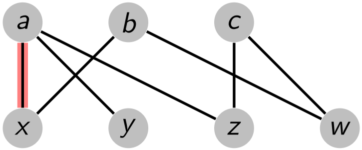

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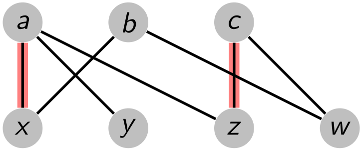

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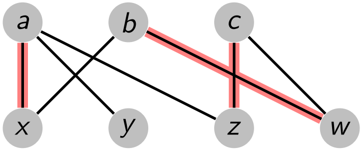

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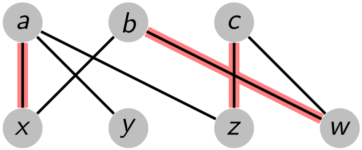
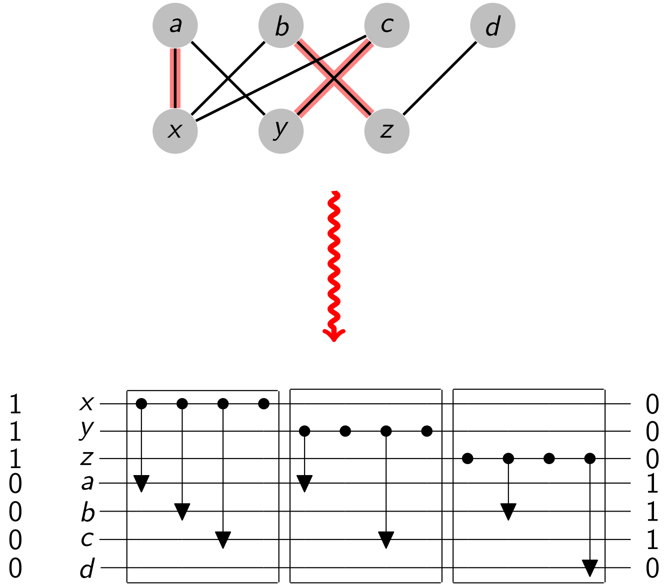


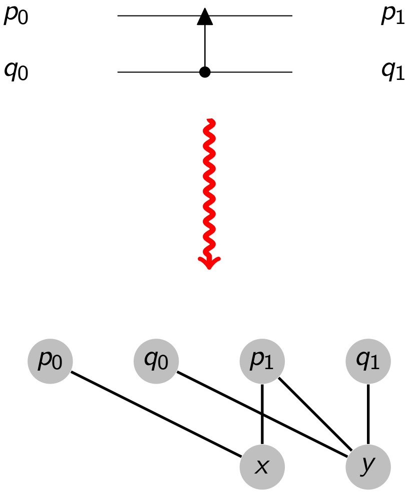
Lex-first maximal matching problem (decision)

Is a given edge $\{u, v\}$ in the lex-first maximal matching of G?

Reducing lex-first maximal matching to CCV



Reducing Ccv to lex-first maximal matching



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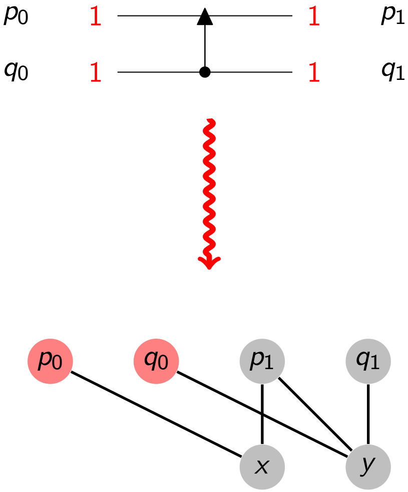

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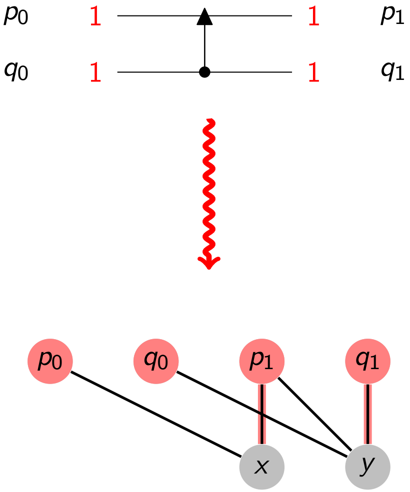

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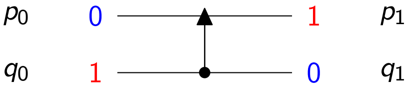

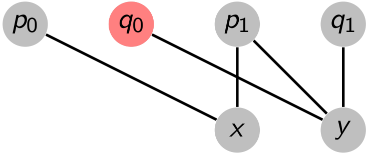

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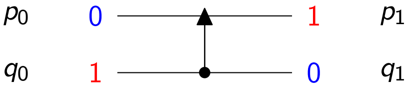

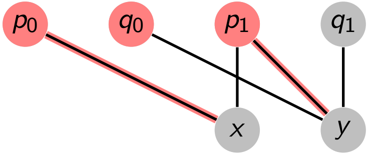

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Two-sorted language $\mathcal{L}_A^2$ (Zambella '96)

Vocabulary $\mathcal{L}_A^2 = [0, 1, +, \cdot, |\ |;\ \in, \leq, =_1, =_2]$

- Standard model $\mathbb{N}_2 = \langle \mathbb{N}, \text{finite subsets of } \mathbb{N} \rangle$
- $0, 1, +, \cdot, \leq, =$ have usual meaning over $\mathbb{N}$
- $|X|$ = length of X
- Set membership $y \in X$

- "number" variables $x, y, z, \ldots$ (range over $\mathbb{N}$)
- "string" variables $X, Y, Z, \ldots$ (range over finite subsets of $\mathbb{N}$)
- Number terms are built from $x, y, z, \ldots, 0, 1, +, \cdot$ and $|X|, |Y|, |Z|, \ldots$
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Note

The natural inputs for Turing machines and circuits are finite strings.

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Definition (Σ_0^B formula)

1. All the number quantifiers are bounded.
2. No string quantifiers (free string variables are allowed)

Two-sorted complexity classes

A two-sorted complexity class consists of relations $R(\vec{x}, \vec{X})$, where

- $\vec{x}$ are number arguments (in unary) and $\vec{X}$ are string arguments

Definition (Two-sorted AC^0)

A relation $R(\vec{x}, \vec{X})$ is in AC^0 iff some alternating Turing machine accepts R in time $\mathcal{O}(\log n)$ with a constant number of alternations.

Σ_0^B-Representation Theorem [Zambella '96, Cook-Nguyen]

$R(\vec{x}, \vec{X})$ is in AC^0 iff it is represented by a Σ_0^B-formula $\varphi(\vec{x}, \vec{X})$.

Useful consequences

1. Don't need to work with uniform circuit families or alternating Turing machines when defining AC^0 functions or relations.
2. Useful when working with AC^0-reductions

The theory V^0 for AC^0 reasoning

The axioms of V^0

1. 2-**BASIC axioms**: essentially the axioms of Robinson arithmetic plus
   - the defining axioms for $\leq$ and the string length function $|\ |$
   - the axiom of extensionality for finite sets (bit strings).
2. Σ_0^B-**COMP** (Comprehension): for every Σ_0^B-formula $\varphi(z)$ without X,
$$\exists X \leq y \, \forall z < y \big(X(z) \leftrightarrow \varphi(z)\big)$$

Theorem

1. Σ_0^B-**IND:** $\big[\varphi(0) \wedge \forall x\big(\varphi(x) \rightarrow \varphi(x+1)\big)\big] \rightarrow \forall x \varphi(x)$, *where* $\varphi \in \Sigma_0^B$.
2. *The provably total functions in* V^0 *are precisely* FAC^0.

Note: Theories, developed using Cook-Nguyen method, extend V^0.

The theory VCC^* for CC^*

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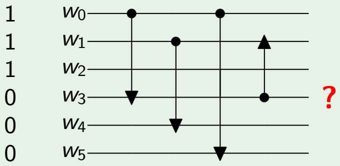
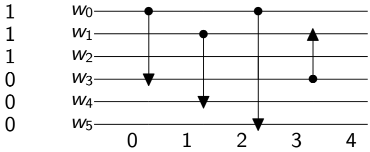


Recall that $\mathsf{CC}^* = \{$decision problems AC^0 oracle-reducible to $\textsc{Ccv}\}$

The two-sorted theory VCC^* [using the Cook-Nguyen method]

- VCC^* has vocabulary $\mathcal{L}_A^2$
- Axiom of VCC^* = Axiom of V^0 + one additional axiom asserting the existence of a solution to the $\textsc{Ccv}$ problem.

Asserting the existence of a solution to Ccv



- X encodes a comparator circuit with m wires and n gates
- Y encodes the input sequence
- Z is an $(n+1) \times m$ matrix, where column i of Z encodes values layer i

The following Σ_0^B formula $\delta_{\mathsf{CCV}}(m, n, X, Y, Z)$ states that Z encodes the correct values of all the layers of the Ccv instance encoded in X and Y:

$$\forall k < m\big(Y(k) \leftrightarrow Z(0,k)\big) \land \forall i < n \forall x < m \forall y < m,$$

$$(X)^i = \langle x, y\rangle \to \left[\begin{array}{ll} & Z(i+1,x) \leftrightarrow \big(Z(i,x) \land Z(i,y)\big) \\ \land & Z(i+1,y) \leftrightarrow \big(Z(i,x) \lor Z(i,y)\big) \\ \land & \forall j < m\Big[(j \neq x \land j \neq y) \to \big(Z(i+1,j) \leftrightarrow Z(i,j)\big)\Big] \end{array}\right]$$

$$\mathsf{VCC}^* = \mathsf{V}^0 + \exists Z \leq \langle m, n+1\rangle + 1,\ \delta_{\mathsf{CCV}}(m, n, X, Y, Z)$$

Conclusion

Summary

1. Introduce the new complexity classes CC and CC^*, which are AC^0-many-one-closure and AC^0-oracle-closure of Ccv respectively.

$$\boxed{\mathsf{NC}^1 \subseteq \mathsf{NL} \subseteq \mathsf{CC} \subseteq \mathsf{CC}^{\mathsf{Subr}} \subseteq \mathsf{CC}^* \subseteq \mathsf{P}}$$

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4. Sharpen and simplify Subramanian's results: we show the following problems are CC-complete (under AC^0-many-one-reduction):
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   - stable-marriage (man-opt, woman-opt and search version)
   - three-valued Ccv (useful when showing the completeness of stable marriage)

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Open Problems

1. $CC = CC^{Subr} = CC^*$?
2. Do universal comparator circuits exist?
3. $CC^* = P$?
4. Do the complete problems in CC have NC or RNC algorithms?