Codifying Probabilities with The Last Full Measure

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November 1, 2023

Introduction

Motivating Example

Laplace's Rule of Succession

Let $n, k \in \mathbb{N}$, $0 \leq k \leq n$, be given/fixed.

1. pick r from $\mathrm{Uniform}(0, 1)$
2. pick c from $\mathrm{Binomial}(n, r)$
3. on condition / restrict to $c = k$

Expected value of $r = (k + 1)/(n + 2)$

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(Finishing what 3Blue1Brown started: probabilities of probabilities part 1, part 2.)

Programming Probabilities

Random Number Generators

$\mathrm{RNG}\,X$ is set/type of programs that output a random X element (probability depends on program)

$\mathrm{Uniform} : \mathbb{R} \times \mathbb{R} \to \mathrm{RNG}\,\mathbb{R}$
$\mathrm{Uniform}(a, b)$ picks real number from $[a, b]$ evenly.

$\mathrm{Binomial} : \mathbb{N} \times [0, 1] \to \mathrm{RNG}\,\mathbb{N}$
$\mathrm{Binomial}(n, r)$ tosses coin n times, head probability r, count heads.

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$\mathrm{Unit}_X : X \to \mathrm{RNG}\, X$
$\mathrm{Unit}_X(x)$ Always outputs x. "Rare desert of determinism in vast oasis of randomization." I omit subscript X if inferrable.

Chaining

Chaining operator “bind”, “then”, “flat-map”
$\gg= \,:\, \mathrm{RNG}\,X \times (X \to \mathrm{RNG}\,Y) \to \mathrm{RNG}\,Y$

$g \gg= k$ passes output of g to parametrized RNG k

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Example:

1. pick r from $\mathrm{Uniform}(0,1)$
2. pick c from $\mathrm{Binomial}(n,r)$
3. output (r,c)

$\mathrm{Uniform}(0,1) \gg= (\lambda r \cdot \mathrm{Binomial}(n,r) \gg= (\lambda c \cdot \mathrm{Unit}(r,c)))$
or
$\mathrm{Uniform}(0,1) \gg= \lambda r \cdot \mathrm{Binomial}(n,r) \gg= \lambda c \cdot \mathrm{Unit}(r,c)$

Restriction/Conditional

Restriction operator
$| : \mathrm{RNG}\, X \times (X \to \mathbb{B}) \to \mathrm{RNG}\, X$
($\mathbb{B}$ = {false, true})

$g \mid pred$ = restrict g to when $pred$ is true.

(Rejection sampling: Keep retrying g until output satisfies $pred$.)

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Example:

1. pick x from $\mathrm{Uniform}(0, 1)$
2. on condition $x > 0.3$
3. (output x)

$\mathrm{Uniform}(0, 1) \mid (\lambda x \cdot x > 0.3)$

Express as Program

Define

$$
\begin{aligned}
g &= \text{Uniform}(0,1) \gg= \lambda r \cdot \text{Binomial}(n,r) \gg= \lambda c \cdot \text{Unit}(r,c) \\
g' &= g \mid (\lambda(r,c) \cdot c = k) \\
\mu &= g' \gg= \lambda(r,c) \cdot \text{Unit}(r)
\end{aligned}
$$

Then μ generates an r as prescribed.

Re-run many times to approximate distribution and expected value.

(Actual Haskell code modulo syntax.)

Re-read Program as Probability Measure

There is also a measure-theory reading of

$$\begin{aligned} g &= \mathrm{Uniform}(0,1) \gg\!= \lambda r \cdot \mathrm{Binomial}(n,r) \gg\!= \lambda c \cdot \mathrm{Unit}(r,c) \\ g' &= g \,|\, (\lambda(r,c) \cdot c = k) \\ \mu &= g' \gg\!= \lambda(r,c) \cdot \mathrm{Unit}(r) \end{aligned}$$

Then μ is the probability distribution/measure of the r in question.

Can find expected value and pdf.

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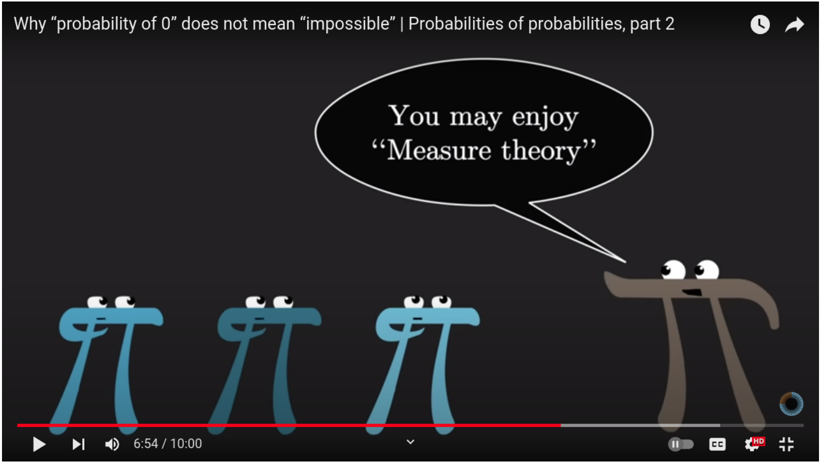
"**Codifying probability** with the last full **measure**."


Why “probability of 0” does not mean “impossible” | Probabilities of probabilities, part 2
You may enjoy
“Measure theory”
6:54 / 10:00

Probability Measures

Measure Theory: Motivation 1/2

Integration by Lateral Thinking. Literally.

Riemann integral: Pixelate the x-axis.

$$\sum_i f(x_i) \times \mathrm{length}[x_i, x_i + \epsilon)$$

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$$\sum_i y_i \times \text{length}\left(f^{-1}[y_i, y_i + \epsilon)\right)$$

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$$\sum_i y_i \times \text{length}\left(f^{-1}[y_i, y_i + \epsilon)\right)$$

Need “length” for fairly general subsets. Require:

$$\text{length}(\bigcup_i A_i) = \sum_i \text{length}(A_i) \quad \text{(countable disjoint union)}$$

Turns out problematic for all subsets. Settle for large enough family closed under complement, countable union.

Measure Theory: Motivation 2/2

Probability Theory

Sample space Ω.

$\Pr$: subsets of $\Omega \to [0, 1]$

Require:

$$\Pr(\bigcup_i A_i) = \sum_i \Pr(A_i) \quad \text{(countable disjoint union)}$$

Hmm déjà vu...

Measure Theory: Motivation 2/2

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Moreover, expected value of $f : \Omega \to \mathbb{R}$ = use $\Pr$ for "length" in Lebesgue integration!

$$\sum_i y_i \times \Pr\left(f^{-1}[y_i, y_i + \epsilon)\right)$$

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$$\sum_i y_i \times \Pr\left(f^{-1}[y_i, y_i + \epsilon)\right)$$

Riemann may not work: Ω may not even have "intervals".

Measurable Space; σ-Algebra

Data of a measurable space:

- Set of points X.
- σ-algebra $\mathfrak{M}$ or $\mathfrak{M}_X$, $\subseteq \wp(X)$, closed under:
  - owning X, owning $\emptyset$
  - complement, countable union
  - (corollary: also subtraction, countable intersection)

  Members “measurable subsets”.

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- $X = \mathbb{R}$, $\mathfrak{M}$ Borel algebra: smallest σ-algebra owning open sets.

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- $X = \mathbb{R}$, $\mathfrak{M}$ Borel algebra: smallest σ-algebra owning open sets.
- $X = \mathbb{R}$, $\mathfrak{M}$ Lebesgue-measurable subsets (appendix). Larger than Borel algebra. Default.

Measures; Probability Measures

[Positive] Measure μ over measurable space $(X, \mathfrak{M})$:

- $\mu : \mathfrak{M} \to [0, \infty]$
- $\mu(\bigcup_i A_i) = \sum_i \mu(A_i)$ for countable disjoint union
  "countably additive"
- (corollary: $\mu(\emptyset) = 0$)
- (corollary: if $A \subseteq B$ then $\mu(A) \leq \mu(B)$)

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Probability measure a.k.a. distribution: Furthermore:

- $\mu(X) = 1$
- (corollary: $\mu : \mathfrak{M} \to [0, 1]$)

Define ΠX = set of probability measures over $(X, \mathfrak{M}_X)$.

Discrete Measures

Counting measure: X countable set, $\mathfrak{M} = \wp(X)$
$\# : \wp(X) \to \mathbb{N} \cup \{\infty\} \subseteq [0, \infty]$

$$\#(A) = \begin{cases} |A| & A \text{ finite} \\ \infty & A \text{ infinite} \end{cases}$$

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Binomial distribution: $X = \mathbb{N}$ (for simplicity), $\mathfrak{M} = \wp(X)$
$\mathrm{Binomial} : \mathbb{N} \times [0, 1] \to \Pi\, \mathbb{N}$

$$\mathrm{Binomial}(n, r)(A) = \sum_{\substack{i \in A \\ 0 \leq i \leq n}} \binom{n}{i} r^i (1 - r)^{n-i}$$

Continuous Measures

Lebesgue measure: $X = \mathbb{R}$, $\mathfrak{M}$ Lebegue-measurable subsets.
$m : \mathfrak{M} \to [0, \infty]$
$m([a, b]) = b - a$, same for $(a, b]$ etc. Full defn in appendix.
Default.

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Other continuous distributions mentioned later after integration.

The Unit Measure

$\mathrm{Unit}_X : X \to \Pi X$

$$\mathrm{Unit}_X(x)(A) = \chi_A(x) = \begin{cases} 0 & x \notin A \\ 1 & x \in A \end{cases}$$

(χ_A characteristic function of set A)

Deterministic corner case.

X can be discrete or continuous or any measurable space.

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X can be discrete or continuous or any measurable space.

In the continuous case, no probability density function.

Measurable Functions

We will only integrate functions that satisfy:

$f : X \to [-\infty, \infty]$ measurable function iff any of:

- for all open B, $f^{-1}(B) \in \mathfrak{M}_X$
- for all $y \in \mathbb{R}$, $\{x \mid f(x) > y\} \in \mathfrak{M}_X$
- or $\geq$, or $<$, or $\leq$

Motivation: Need $\mu\left(f^{-1}[y, y+\epsilon)\right)$, makes sense for $\mathfrak{M}_X$ only.

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Motivation: Need $\mu\left(f^{-1}[y, y + \epsilon)\right)$, makes sense for $\mathfrak{M}_X$ only.

Easy: $f : \mathbb{N} \to [-\infty, \infty]$ is measurable using $\mathfrak{M} = \wp(\mathbb{N})$.

Theorem: Piecewise continuous $f : \mathbb{R} \to \mathbb{R}$ is measurable using Lebesgue-measurable subsets.

Lebesgue Integration: Idea

Notation: $\int f\, d\mu$ integrates f over all of X using measure μ.

Assume $f \geq 0$ for now. Approximate from below by simple functions (finite range).

Example: $X = [-4, 4]$, $f(x) = x^2$, an approximation is

$$s(x) = 4 \cdot \chi_{(-3,-2]\cup[2,3)}(x) + 9 \cdot \chi_{[-4,-3]\cup[3,4]}(x)$$

"Clearly"

$$\int s\, d\mu = 4 \cdot \mu((-3, -2] \cup [2, 3)) + 9 \cdot \mu([-4, -3] \cup [3, 4])$$

Take supremum over all approximations.

For general f, split into positive and negative parts, both treatable as above.

Lebesgue Integration

Simple functions (finite range):

$$\int \sum_{i=1}^{n} a_i \, \chi_{E_i} \, d\mu = \sum_{i=1}^{n} a_i \, \mu(E_i) \quad (a_i \in \mathbb{R}, \, E_i \in \mathfrak{M})$$

Extend to non-negative $f : X \to [0, \infty]$

$$\int f \, d\mu = \sup\{\int s \, d\mu \mid s \text{ simple}, 0 \leq s \leq f\}$$

Extend to full range $f : X \to [-\infty, \infty]$

$$\int f \, d\mu = \left(\int \max(0, f) \, d\mu\right) - \left(\int -\min(f, 0) \, d\mu\right)$$

assuming not $\infty - \infty$.

Lebesgue Integration

Notation: $\int_A f\, d\mu$ is over arbitrary $A \in \mathfrak{M}$, instead of all X.

Two equivalent treatments:

- Revise the definitions, change $\mu(E_i)$ to $\mu(E_i \cap A)$.
- Just use $\int f \chi_A\, d\mu$

Discrete Integration

X countable set, $\mathfrak{M} = \wp(X)$, counting measure:

$$\int f \, d\# = \sum_{i \in X} f(i)$$

(assume $f \geq 0$ or else absolute convergence or other conditions)

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(assume $f \geq 0$ or else absolute convergence or other conditions)

Binomial expressible as integral $d\#$:

$$\mathrm{Binomial}(n, r)(A) = \int_A \lambda i \cdot \binom{n}{i} r^i (1-r)^{n-i} \, d\#$$

Hence $\lambda i \cdot \binom{n}{i} r^i (1-r)^{n-i}$ is probability mass function.

$$\int f \, d(\mathrm{Binomial}(n, r)) = \int \lambda i \cdot f(i) \binom{n}{i} r^i (1-r)^{n-i} \, d\#$$

Continuous Integration (Sorry!)

$\int f\,dm$ default for $X = \mathbb{R}$ or subspace. Riemann toolbox reusable because:

Theorem: For Riemann-integrable $f : [a, b] \to \mathbb{R}$,

$$\int_{[a,b]} f\,dm = \text{Riemann} \int_a^b f(x)\,dx$$

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Most continuous distributions have probability density functions:

$$\mu(A) = \int_A \text{pdf}\, dm$$

$$\int f\, d\mu = \int f\text{pdf}\, dm$$

Unit Integration

Sifting property:

$$\int f \, d(\mathrm{Unit}(x)) = f(x)$$

Integral well defined (and beautiful), even in continuous case, even if no pdf. The power of measure theory!

Chaining Probability Measures

Chaining operator “bind”, “then”, “flat-map”
$\gg= \, : \Pi X \times (X \to \Pi Y) \to \Pi Y$

Intuition for $(\mu \gg= k)(B)$: For each $x \in X$, $k(x) \in \Pi Y$, $k(x)(B)$ is probability of B. Average over X according to μ. (Total probability).

Likewise for expected values.

$$(\mu \gg= k)(B) = \int \lambda x \cdot k(x)(B) \, d\mu$$

$$\int f \, d(\mu \gg= k) = \int \left(\lambda x \cdot \int f \, d(k(x)) \right) d\mu$$

(Require $k : X \to \Pi Y$ measurable function. Boils down to $(\lambda x \cdot k(x)(B)) : X \to [0, 1]$ measurable function for all $B \in \mathfrak{M}_Y$.)

Interlude

Happy Belated Halloween! 👻

$(\Pi, \text{Unit}, \gg=)$ is a programmer-friendly version of…

The Giry monad! Due to Michèle Giry. More in appendix.

Restricting Probability Measures

Restriction operator
$| : \Pi X \times (X \to \mathbb{B}) \to \Pi X$

Conditional probability.

$$(\mu \,|\, pred)(A) = \mu(A \cap B)/\mu(B)$$
$$\int f \, d(\mu \,|\, pred) = \frac{1}{\mu(B)} \int f \chi_B \; d\mu$$

where $B = \{x \mid pred(x)\}$, assuming $\in \mathfrak{M}$.

(One may say: $pred$ is a measurable predicate.)

Re-read Program as Probability Measure

Define

$$
\begin{aligned}
g &= \text{Uniform}(0,1) \gg\!= \lambda r\cdot \text{Binomial}(n,r) \gg\!= \lambda c\cdot \text{Unit}(r,c) \\
g' &= g \,|\, (\lambda(r,c)\cdot c = k) \\
\mu &= g' \gg\!= \lambda(r,c)\cdot \text{Unit}(r)
\end{aligned}
$$

Then μ is the probability measure of the r in question.

$$
\begin{aligned}
\text{expected} &= \int \lambda r\cdot r \, d\mu \\
&= (n+1) \int_0^1 r\binom{n}{k} r^k (1-r)^{n-k} \, dr \qquad \text{appendix} \\
&= \frac{k+1}{n+2} \\
\text{pdf}(r) &= (n+1)\binom{n}{k} r^k (1-r)^{n-k}
\end{aligned}
$$

Conclusion

Conclusion

Formal constructs for expressing probabilistic models.

Program interpretation gives executable samplers.

Meaure-theory interpretation gives resulting distributions.

Works the same way for discrete, continuous, hybrid, nested.

Bibliography

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Appendix

Lebesgue Measure And Measurable Subsets

The following is well-defined for all $A \in \wp(\mathbb{R})$; denoted $m^*(A)$:

- let $\{I_n\}$ be sequence of disjoint open intervals, $\bigcup_n I_n \supseteq A$
- $\sum_n \mathrm{length}(I_n)$ uncontroversial
- take infinum over all possibilities

Only issue: Not countably additive for some subsets.

Carathéodory's solution: Restrict to

$$\mathfrak{M} = \{A \mid \forall B \cdot m^*(B) = m^*(B \cap A) + m^*(B - A)\}$$

Adopted for Lebesgue-measurable subsets.

Then define Lebesgue measure $m = m^*|_{\mathfrak{M}}$

The Giry Monad 1/2

Prologue: Generalize "measurable functions" from $X \to [-\infty, \infty]$ to $X \to Y$, with Y any measurable space:
$h : X \to Y$ measurable iff $\forall B \in \mathfrak{M}_Y \cdot h^{-1}(B) \in \mathfrak{M}_X$

Π as endofunctor (on category of measurable spaces):

Object map: $\Pi\, X$ = set of probability measures over $(X, \mathfrak{M}_X)$.
σ-algebra: Smallest s.t. for all $A \in \mathfrak{M}_X$, $(\lambda\mu \cdot \mu(A)) : \Pi\, X \to [0, 1]$ measurable function.

Morphism map: For measurable $h : X \to Y$, $\Pi\, h : \Pi\, X \to \Pi\, Y$

$$(\Pi\, h)(\mu)(B) = \mu(h^{-1}(B))$$
$$\int f\, d((\Pi\, h)(\mu)) = \int f \circ h\, d\mu$$

The Giry Monad 2/2

Π as monad:

$\mathrm{Unit}_X : X \to \Pi X$ as defined earlier.

Multiplication $\mathrm{Flat}_X : \Pi(\Pi X) \to \Pi X$

$$\mathrm{Flat}(\mu)(A) = \int \lambda \nu \cdot \nu(A)\, d\mu$$

$$\int f\, d(\mathrm{Flat}(\mu)) = \int \left(\lambda \nu \cdot \int f\, d\nu \right) d\mu$$

Flattens “probability of probabilities” to total probability.

Then define $\mu \gg= k = \mathrm{Flat}((\Pi\, k)(\mu))$.

Detailed Calculation 1/2

Let $B = \{(r, c) \mid c = k\} = [0, 1] \times \{k\}$.

$$\int f\,dg = \int \left(\lambda r \cdot \int \left(\lambda c \cdot \int f\,d(\mathrm{Unit}(r, c))\right) d(\mathrm{Binomial}(n, r))\right)$$
$$d(\mathrm{Uniform}(0, 1))$$
$$= \int_0^1 \sum_{c=0}^{n} f(r, c)\binom{n}{c} r^c (1 - r)^{n-c}\,dr$$
$$g(B) = \int \chi_B\,dg$$
$$= \int_0^1 \binom{n}{k} r^k (1 - r)^{n-k}\,dr$$
$$= \binom{n}{k}\frac{k!(n-k)!}{(n+1)!} \qquad \text{(Beta functions)}$$
$$= \frac{1}{n+1}$$

Detailed Calculation 2/2

$$
\begin{aligned}
\int \lambda r \cdot r \, d\mu &= \int \left(\lambda(r,c) \cdot \int \lambda r \cdot r \, d(\mathrm{Unit}(r)) \right) dg' \\
&= \int (\lambda(r,c) \cdot r) \; dg' \\
&= \frac{1}{g(B)} \int (\lambda(r,c) \cdot r \chi_B(r,c)) \; dg \\
&= \frac{1}{g(B)} \int_0^1 r \binom{n}{k} r^k (1-r)^{n-k} \, dr \\
&= (n+1) \binom{n}{k} \frac{(k+1)!(n-k)!}{(n+2)!} \qquad \text{(Beta functions)} \\
&= \frac{k+1}{n+2}
\end{aligned}
$$