Problem Set 3

Graphical Models

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1 Convex Belief Propagation

In most graphs with circles, entropy term $H(p)$ is computationally intractable. The salification of simplex $\mathcal{P}$ is also intractable. The most widely used tractable approximation of this optimization problem are based on (1) decomposition of $H(p)$ into local entropy terms, and (2) approximation of $\mathcal{P}$ by simpler convex outer bounds, such as local marginal consistency constraints. The true marginal distributions $p(x_i)$ and $p(x_i, x_j)$ are replaced by "belief" $b_i(x_i)$ and $b_{i,j}(x_i, x_j)$. The entropy approximation $\hat{H}(b)$ has the form:

$$\sum_{(i,j)\in E} c_{i,j} H(b_{i,j}) + \sum_i c_i H(b_i)$$

The probability simplex is replaced by a local polytope $\mathcal{L}(b)$ defined below:

$$\mathcal{L}(b) = \begin{cases} \sum_{x_j} b_{i,j}(x_i, x_j) = b_i(x_i), \forall (i,j) \in E \\ b_{i,j}(x_i, x_j) \geq 0, \sum_{x_i, x_j} b_{i,j}(x_i, x_j) = 1 \end{cases}$$

For trees, the setting of $c_{i,j} = 1$ and $c_i = 1 - d_i$ where d_i is the degree of node i, is exact and known as the Bethe entropy:

$$H_{bethe}(b) = \sum_{(i,j)\in E} H(b_{i,j}) + \sum_i (1 - d_i) H(b_i)$$

Moreover, when the underlying graph is a tree, the local polytope is equal to the probability simplex or marginal polytope. As a result, the Bethe free energy problem is both exact and convex for tree-structured graph.

For general graphs with cycles, the Bethe entropy is an approximation of true entropy. Also, the local polytope is an outer bounds of marginal polytope. So there is not guarantee of the minimizer of Bethe free energy problem. From the optimization point of view, the Bethe entropy is no longer convex for graphs with cycles. As a result, the fixed point of sum-product algorithm is only a local minima of the optimization problem.

A popular generalization of the Bethe free energy is known as the fractional free energy. By some clever ways of setting counting numbers c, $\hat{H}$ could be concave. A set of sufficient conditions of concave entropy approximation is $c_i, c_{ij} > 0$.

2 Problem 1

Derive the belief propagation (message passing) algorithm as fixed points for the Bethe entropy, for a chain graph with 4 vertices, namely $1 \to 2 \to 3 \to 4$.

3 Problem 2

Derive a belief propagation (message passing) algorithm (Convex BP) the convex free energy approximation with given $c_i, c_{i,j} > 0$.

4 Programming Assignment

Compare Convex BP with Loopy belief propagation (sum-product) algorithm for grid graphs In this problem, we use the same graph model as the one in exercise 2.
1) Bethe free energy works very well in practice. Run Convex BP with the positive counting numbers, and compare the marginal errors of Convex BP and Loopy BP for each $b_i(x_i)$.